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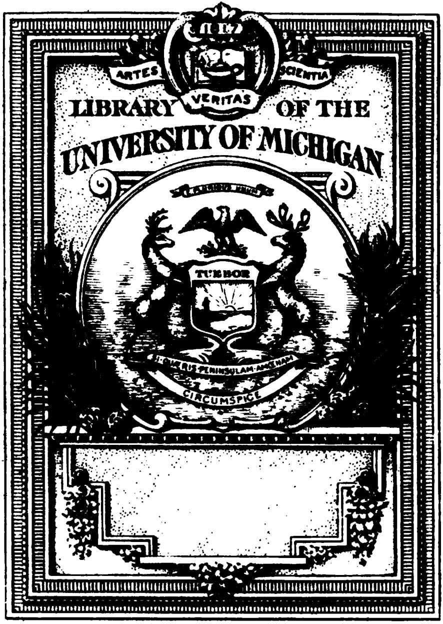

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A TREATISE

ON

ELEMENTARY TRIGONOMETRY.

A TREATISE

ON

ELEMENTARY TRIGONOMETRY

W. W. BEMAN,
11 South 5th St.,
ANN ARBOR, MICH.

BY THE

REV. J.ohn B.ascombe LOCK, M.A. 1849-

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London:
MACMILLAN & CO.
1885

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STEREOTYPED EDITION REVISED 1883.

Cambridge:
PRINTED BY C. J. CLAY, M.A. & SON,
AT THE UNIVERSITY PRESS.

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PREFACE.

The present Work on Elementary Trigonometry contains that part of the subject which can conveniently be explained without the use of infinite series. It is intended either for class-teaching or for private study. Accordingly the Examples are numerous and for the most part easy. Those which are not original have been selected from the Cambridge and Army Examination Papers of the last few years.

The Miscellaneous Examples are somewhat more difficult, and should in most cases be postponed until the student reads the subject for the second time.

The order in which the chapters are read may be varied at the discretion of the Teacher.

The last three Chapters are not required in the Previous Examination of Candidates for Honours at Cambridge.

J. B. L.

Eton,
March, 1882.

In the Second Edition a short course has been indicated for the use of Students who wish to read the Solution of Triangles as early as possible. Such

Students are advised to omit every article that is marked with an asterisk.

A double asterisk has been placed before those articles which should be omitted by all Students until they are reading the subject for the second time.

At the suggestion of Mr H. S. Hall, Master at Clifton College, an Appendix has been added containing examples of a kind which occurs frequently in Examination Papers and which illustrates the principles explained in Chapters XVI. and XVII.

Gonville and Caius College,
August, 1884.

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CHAPTER I.

ON MEASUREMENT.

1. IT is usual to say that we have measured any concrete quantity, when we have found out *how many* times it contains some familiar quantity of the same kind.

We say for example, that we have measured a line, when we have found out *how many* feet it contains. We say that we have measured a field, when we have found out *how many* acres or how many square yards it contains.

2. To know the measurement of any quantity then, we must have two things. First, we must have a *unit*, or standard of reference, of the *same kind* as the thing measured. Secondly, we must have the *measure*, or the *number of times* the thing measured contains the unit, or standard quantity.

3. Hence, the **measure** of a quantity is the **number,** and the **unit** is the **concrete quantity,** by means of which it is measured.

Example 1. A line contains 261 feet. Here the *measure* or number is 261 and the *unit* a foot.

Example 2. What is the measure of $2\frac{1}{2}$ miles when a yard is the unit?

$$2\tfrac{1}{2} \text{ miles} = \tfrac{5}{2} \times 1760 \text{ yards}, \\ = 4400 \text{ yards} = 4400 \times 1 \text{ yard},$$

therefore the measure is 4400 when a yard is the unit.

Example 3. What is the unit when the measure of a field of 10 acres is 242?

$$10 \text{ acres} = 242 \times \frac{10 \text{ acres}}{242},$$

$$\therefore \text{ the unit is } \frac{10 \text{ acres}}{242} = 200 \text{ square yards.}$$

Example 4. If the unit be a yards, what is the measure of b miles?

$$b \text{ miles} = b \times 1760 \text{ yards},$$

$$\therefore\ b \text{ miles} = \frac{b \times 1760}{a} \times a \text{ yards},$$

$$\therefore \text{ the measure required is } \frac{b \times 1760}{a}.$$

EXAMPLES. I.

(1) What is the measure of 1 mile when a chain of 66 feet is the unit?

(2) What is the measure of an acre when a square whose side is 22 yards is the unit?

(3) What is the measure of a ton when a weight of 10 stone is the unit?

(4) The length of an Atlantic cable is 2300 miles and the length of the cable from England to France is 21 miles. Express the length of the first in terms of the second as unit.

(5) The measure of a certain field is 22 and the unit 1100 square yards: express the area of the field in acres.

(6) Find the measure of a miles when b yards is the unit.

(7) The measure of a certain distance is a when the unit is c feet. Express the distance in yards.

(8) A certain sum of money has for its measures 24, 240, 960 when three different coins are units respectively. If the first coin is half a sovereign, what are the others?

* 4. The measure of a quantity is the number of times which that quantity contains the unit.

We may express the same thing in different ways,

(i) The *measure* of a quantity is the **ratio** of that quantity to the unit.

(ii) The *measure* of a quantity is the **fraction** that the quantity is of the unit.

* 5. This last statement is in the language of Arithmetic, and the word 'fraction' is to include whole numbers, and improper fractions.

* 6. The following are therefore different forms of the same question:

(i) What is the measure of 4 miles when 66 feet is the unit?

(ii) How many times does 4 miles contain 66 feet?

(iii) What is the ratio of 4 miles to 66 feet?

(iv) What fraction of 66 feet is 4 miles?

Example (i) What is the measure of a yards when b feet is the unit?

$$a \text{ yards} = 3a \text{ feet} = \frac{3a}{b} \times b \text{ feet},$$

$$\therefore \text{ the measure is } \frac{3a}{b}.$$

(ii) How many times does a yards contain b feet?

$$\text{As in (i), } a \text{ yards} = \frac{3a}{b} \times b \text{ feet}.$$

Answer. $\frac{3a}{b}$ times.

(iii) What is the ratio of a yards to b feet?

$$\text{As in (i), } a \text{ yards} = \frac{3a}{b} \times b \text{ feet},$$

$$\therefore \frac{a \text{ yards}}{b \text{ feet}} = \frac{3a}{b},$$

$$\therefore \text{ the required ratio is } \frac{3a}{b}.$$

(iv) What fraction is a yards of b feet?

$$\text{As in (iii), } \frac{a \text{ yards}}{b \text{ feet}} = \frac{3a}{b},$$

$$\therefore \text{ the required fraction is } \frac{3a}{b}.$$

* EXAMPLES. II.

(1) The ratio of the area of one field to that of another is 20 : 1, and the area of the first is half a square mile. Find the number of square yards in the second.

(2) The ratio of the heights of two persons is 9 : 8, and the height of the second is 5 ft. 4 in. What is the height of the first?

(3) The measure of a field with 3 acres for unit is 65. Find the ratio of the field to an acre.

(4) One field contains a second of $2\frac{1}{2}$ acres, $6\frac{3}{4}$ times.
What is the measure of the first field in terms of the second?
What is the ratio of the first field to the second?
Express the first as the fraction of the second.

(5) A certain weight is 3·125 of a ton.
What is its measure in terms of 4 cwt.?
How many times does it contain 4 cwt.?
What is its ratio to 4 cwt.?
What fraction is it of 4 cwt.?

(6) The ratio of a certain sum of money to 3 guineas is $\frac{22}{7}$.
Find its measure in terms of one pound.
Find the unit when its measure is 22.

(7) What is the measure of a miles when b chains is the unit?
How many times do a miles contain c chains?
What is the ratio of a miles to d chains?
What fraction is a miles of k chains?

(8) What is the unit when the measure of £20 is $\frac{400}{21}$?

£25 contains a certain sum $\frac{500}{21}$ times. What is that sum?

The ratio of £30 to a certain sum is $\frac{600}{21}$. What is that sum?

The fraction which £10 is of a certain sum is $\frac{200}{21}$. What is that sum?

7. It is explained in Arithmetic, in the application of square measure, that the measure of the area of a rectangle is found in terms of a square unit, by multiplying together the measures of the sides in terms of the corresponding linear unit.

Example. Find in square feet the measure of a square surface whose side is 12 feet.

The area is 12×12 square feet $= 144 \times 1$ square foot,

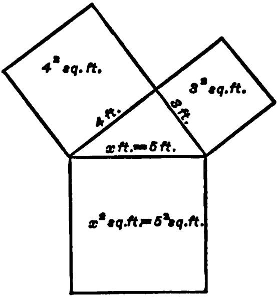
$\therefore$ the measure required is 144.

8. We shall apply this result to Euclid I. 47.

Example 1. The sides containing the right angle of a right-angled triangle are 3 ft. and 4 ft. respectively; find the length of the hypotenuse.

Let x be the number of feet in the hypotenuse.

Then by Euclid I. 47, the square described on the side of x feet = the sum of the squares described on the sides of 3 feet and 4 feet respectively,

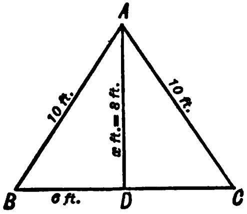


$$\therefore\ x^2 \text{ square feet} = 9 \text{ square feet} + 16 \text{ square feet}$$
$$= 25 \text{ square feet,}$$
$$\therefore\ x^2 = 25,$$
$$\therefore\ x = 5.$$

Therefore the length of the hypotenuse is 5 feet.

Example 2. Find the length of the perpendicular drawn from the vertex to the base of an isosceles triangle whose equal sides are 10 feet each, and whose base is 12 feet.

Let *ABC* be the isosceles triangle such that *AB* is 10 feet, *AC* is 10 feet and *BC* is 12 feet.



Draw *AD* perpendicular to *BC*.

Then because the triangle *ABC* is isosceles *AD* will bisect the base *BC* in *D*; therefore *BD* is 6 feet.

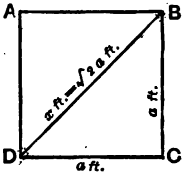
Let *AD* contain x feet.

Then by Euclid I. 47, the square on *AB* = the sum of the squares on *BD* and *AD*.

$$\therefore\ 10^2 \text{ sq. ft.} = 6^2 \text{ sq. ft.} + x^2 \text{ sq. ft.}$$
$$\therefore\ 10^2 = 6^2 + x^2,$$
$$\therefore\ x^2 = 100 - 36,$$
$$\therefore\ x^2 = 64,$$
$$\therefore\ x = 8.$$

Therefore the required length of *AD* is 8 feet.

Example 3. Find the length of the diameter of the square one of whose sides contains a feet.



Let *ABCD* be the square, so that *AB* is a feet, and *AD* is a feet.

Let the diameter *BD* be x feet.

Then the square on *DB* = the sum of the squares on *DA* and *AB*.

$$\therefore\ x^2 \text{ sq. ft.} = a^2 \text{ sq. ft.} + a^2 \text{ sq. ft.}$$

$$\therefore\ x^2 = a^2 + a^2,$$

$$\therefore\ x^2 = 2a^2,$$

$$\therefore\ x = \sqrt{2}\,.\,a.$$

Therefore the required length of the diameter is $\sqrt{2}\,.\,a$ feet.

EXAMPLES. III.

(1) Find the length of the hypotenuse of a right-angled triangle whose sides are 6 feet and 8 feet respectively.

(2) The hypotenuse of a right-angled triangle is 100 yards and one side is 60 yards: find the length of the other side.

(3) One end of a rope 52 feet long is tied to the top of a pole 48 feet high and the other end is fastened to a peg in the ground. If the pole be vertical and the rope tight, find how far the peg is from the foot of the pole.

(4) The houses in a certain street are 40 feet high and the street 30 feet wide: find the length of the ladder which will reach from the top of one of the houses to the opposite side of the street.

(5) A wall 72 feet high is built at one edge of a moat 54 feet wide; how long must scaling ladders be to reach from the other edge of the moat to the top of the wall?

(6) A field is a quarter of a mile long and three-sixteenths of a mile wide: how many cubic yards of gravel would be required to make a path 2 feet wide to join two opposite corners, the depth of the gravel being 2 inches?

(7) The sides of a rectangular field are $4a$ feet and $3a$ feet respectively. Find the length of its diameter.

(8) If the sides of an isosceles triangle be each $13a$ yards and the base $10a$ yards, what is the length of the perpendicular drawn from the vertex to the base?

(9) Show that the perpendicular drawn from the right angle to the hypotenuse in an isosceles right-angled triangle, each of whose equal sides contains a feet, is $\frac{\sqrt{2}}{2} . a$ ft.

(10) If the hypotenuse of a right-angled isosceles triangle be a yards, what is the length of each side?

(11) Show that the perpendicular drawn from an angular point to the opposite side of an equilateral triangle, each of whose sides contains a feet, is $\frac{\sqrt{3}}{2} . a$ ft.

(12) If in an equilateral triangle the length of the perpendicular drawn from an angular point to the opposite side be a feet, what is the length of the side of the triangle?

(13) Find the ratio of the side of a square inscribed in a circle to the diameter of the circle.

(14) Find the distance from the centre of a circle of radius 10 feet, of a chord whose length is 8 feet.

(15) Find the length of a chord of a circle of radius a yards, which is distant b feet from the centre.

(16) The three sides of a right-angled triangle, whose hypotenuse contains $5a$ feet, are in arithmetical progression: prove that the other two sides contain $4a$ feet and $3a$ feet respectively.

* CHAPTER II.

On Incommensurable Quantities.

9. Two numbers are said to be **commensurable** when their ratio can be expressed as an *arithmetical* fraction: that is, as a fraction whose numerator and denominator are both whole numbers.

Example. $4{\cdot}9\dot{3}$ and $81\frac{3}{4}$ are two commensurable numbers.

Their ratio is $4\frac{84}{90} \div 81\frac{3}{4} = \dfrac{444 \times 4}{90 \times 327}$.

10. Two numbers are said to be **incommensurable** when their ratio *cannot* be expressed as an arithmetical fraction.

Example. $\sqrt{2}$ and 1 are two incommensurable numbers. For $\frac{\sqrt{2}}{1}$ cannot be expressed exactly as an arithmetical quantity. So $\sqrt{3}$ and $\sqrt{2}$ are two incommensurable numbers.

11. One number alone is said to be *an incommensurable number* when it is incommensurable with *unity*. So that an incommensurable number cannot be expressed as an arithmetical fraction.

Example. $\sqrt{2}$ and $\sqrt{3}$, and all surd numbers, are incommensurable numbers.

12. Two *quantities* are said to be commensurable when their *measures* referred to a common unit are commensurable.

Example. A mile and a thousand yards are two commensurable quantities. Their measures with a yard for unit are 1760 and 1000; and these are commensurable numbers.

13. Two quantities are said to be incommensurable when their measures referred to a common unit are incommensurable.

Example. The side of a square and its diameter are two incommensurable quantities. For if the side of a square contain a feet, the diameter (see Example 3, p. 7) contains $\sqrt{2} . a$ feet, and therefore the ratio of their measures is $1 : \sqrt{2}$. So that their measures are incommensurable.

14. There is *no practical difficulty* in dealing with incommensurable quantities. We can always find for their measures arithmetical expressions sufficiently accurate for all practical purposes.

15. A little consideration will convince the student that no measurement can in *practice* be made with *absolute* accuracy.

For example: A skilful mechanic is probably satisfied if in measuring some material two or three feet in length the error in his measurement is less than the thirty-second part of an inch. (The thirty-second part of an inch is less than half the height of the smallest letter on this page.) That is to say, he is satisfied if he make no greater error than about a *thousandth part of the whole length* to be measured. He would record such a measurement thus,

$$2 \text{ ft. } 3\tfrac{27}{32} \text{ inches,}$$

which is 891 thirty-second parts of an inch.

16. We will suppose that the length thus measured is the side of a square, and that the workman wishes to know to the same degree of accuracy as his measurement, what is the length of the diameter of the square.

He can find it thus.

The diameter of a square $= \sqrt{2} \times$ its side,

$\therefore$ the diameter of this square $= \sqrt{2} \times 891$ thirty-second parts of an inch.

$$\text{Also } \sqrt{2} = 1{\cdot}414 \text{ nearly,}$$

$$\therefore \sqrt{2} \times 891 = 1{\cdot}414 \times 891$$
$$= 1259{\cdot}8.$$

$\therefore$ the required diameter $= 1260$ thirty-second parts of an inch, nearly.

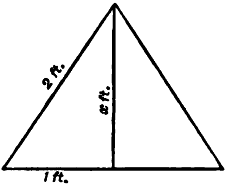
The error being less than one thirty-second part of an inch.

17. The student will be able to see from the above example, that if the value of an incommensurable number is found to 4 figures, a very considerable degree of accuracy is attained. Also that a much greater degree of accuracy is attained for every additional figure.

18. It is no advantage in calculations such as the above to have the value of such quantities as $\sqrt{2}$ calculated to any greater degree of accuracy than the observed measurement. For instance, our calculation being correct as far as the whole numbers are concerned we should gain nothing by using 1·4142 instead of 1·414 for $\sqrt{2}$. This would give us the answer 1260·0 instead of 1259·8; the difference being a fifth of a thirty-second part of an inch, a quantity by hypothesis too small to be of any importance.

Example 1. The side of an equilateral triangle contains 2 feet; find, correct to the ten-thousandth part of a foot, the length of the perpendicular drawn from an angular point to the opposite side.

Here (as in Example 2, p. 6), let the perpendicular contain x feet, then

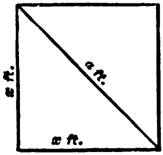


$$x^2 \text{ sq. ft.} = 2^2 \text{ sq. ft.} - 1^2 \text{ sq. ft.},$$
$$\therefore\ x^2 = 4 - 1 = 3,$$
$$\therefore\ x = \sqrt{3},$$
$$= 1{\cdot}7320 \text{ \&c.};$$

$\therefore$ the length of the perpendicular $= 1{\cdot}7320$ feet.

Example 2. Find the length, correct to the ten-thousandth part of a foot, of the side of the square described upon a diameter whose length is a feet.

Let x be the number of feet in each of the sides of the square.



Then x^2 sq. ft. $+ x^2$ sq. ft. $= a^2$ sq. ft.

$$\therefore x^2 + x^2 = a^2,$$

$$\therefore 2x^2 = a^2,$$

$$\therefore \sqrt{2} . x = a,$$

$$\therefore x = \frac{a}{\sqrt{2}} = \frac{a\sqrt{2}}{2}$$

$$= \frac{a}{2} \times (1{\cdot}4142) = a \times {\cdot}7071\ldots,$$

$\therefore$ the length of the perpendicular $= a \times {\cdot}7071\ldots$ feet.

* EXAMPLES. IV.

(1) Find, correct to the thousandth part of a foot, the length of the diameter of a square whose side is 7 feet.

(2) Find, correct to a yard, the length of the diameter of a square whose side is one mile.

(3) Find, correct to the hundredth part of an inch, the hypotenuse of a right-angled triangle whose sides are 3 ft. 6½ in. and 3 ft. 4 in. respectively.

(4) Find, to the nearest inch, the side of a square whose area is 1000 square yards.

(5) Find, correct to the tenth part of a foot, the diameter of a square field whose area is ten acres.

(6) Find, to the nearest inch, the side of a square field whose area is one acre.

(7) Find the height of an equilateral triangle whose side is 10 feet.

(8) Find the height of an equilateral triangle whose side is 5·32 feet.

(9) The top of a table measures 24·6 inches and 41·3 inches along two adjacent sides. What should the diameter measure if the table is rectangular?

(10) Find, to the nearest inch, the diameter of a lawn-tennis court whose length is 78 feet, and breadth 36 feet, supposing that it is properly marked out.

CHAPTER III.

ON THE RELATION BETWEEN THE CIRCUMFERENCE OF A CIRCLE AND ITS DIAMETER.

19. THE circumference of a circle is a line, and therefore it has length.

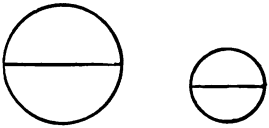
We might imagine the circumference of a circle to consist of a flexible wire; if the circular wire were cut at one point and straightened, we should have a straight line of the same length as the circumference of the circle.

20. A *polygon* is a figure enclosed by any number of straight lines.

21. A *regular* polygon has all its sides equal and all its angles equal.

22. The *perimeter* of a polygon is the sum of its sides.

23. If we have two circles in which the diameter of the first is greater than the diameter of the second, it is evident that the circumference of the first will be greater than the circumference of the second.



24. It seems, therefore, not unlikely that, if the diameter of the first circle be twice that of the second, the circumference of the first will be twice that of the second.

25. And also not unlikely that whatever be the ratio of the circumference of the first circle to its diameter, the same will be the ratio of the circumference of the second circle to its diameter.

This suggests that it is not unlikely that the circumference of a circle $= k$ times its diameter, where k is some *number* which is the same for all circles.

We shall presently prove that this is the case.

26. But although we can prove that

$$\frac{\text{the circumference of a circle}}{\text{its diameter}} = \text{a fixed numerical quantity,}$$

the method of calculating the value of this number is beyond the limits of an elementary treatise.

27. We shall therefore simply state here, what is proved in the Higher Trigonometry,

(i) that this numerical value is incommensurable,

(ii) that it is approximately 3·14159265 &c.

28. When we say that this number is incommensurable we mean (cf. Chapter II.) that its exact value cannot be stated as an *arithmetical* fraction.

It also happens that we have no short *algebraical* expression such as a surd, or combination of surds, which represents it exactly.

So that we have no *numerical* expression whatever, arithmetical nor algebraical, to represent *exactly* the ratio of the circumference of a circle to its diameter.

Hence the universal custom has arisen, of denoting its *exact* value by the letter π.

29. Thus π stands *always* for **the exact value of a certain incommensurable number**, whose approximate value is **3·14159265**, which number is **the ratio of the circumference of any circle to its diameter.**

It cannot be too carefully impressed on the student's memory that π stands for this number 3·14159265...&c., and for nothing else; just as 180 stands for the number one hundred and eighty, and for nothing else.

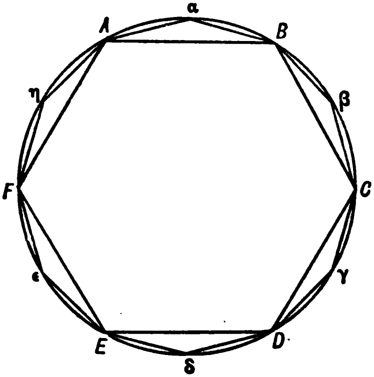
*30. We proceed to prove that the ratio of the circumference of a circle to its diameter is the same for all circles.

The proof depends on the following important principle;

The length of the circumference of a circle is that to which the length of the perimeter of a regular inscribed polygon approaches, as the number of its sides is continually increased.

*31. We take for granted that the *straight* line is the *shortest* line that can join two points.

*32. Let *ABCDEF* be any regular polygon inscribed in a circle.



Then the side AB is shorter than any other line joining AB, so that the side AB is less than the arc AB;

therefore the perimeter of the polygon, viz.

$$AB + BC + CD + DE + EF + FA$$

is less than the sum of the arcs, that is, is less than the circumference of the circle.

Now let each of the *arcs* AB, BC, etc. be bisected in α, β, γ, δ, ϵ, η, and let the lines $A\alpha$, αB, $B\beta$, βC, etc. be joined;

then the figure $A\alpha B\beta C\gamma$ etc. is a regular polygon of twice as many sides as the first polygon.

And since the sides $A\alpha + \alpha B$ are together greater than the side AB,

it follows that the perimeter of the second polygon, viz.

$$A\alpha + \alpha B + B\beta + \beta C + C\gamma + \gamma D + \text{etc.}$$

is greater than the perimeter of the first.

But the perimeter of the second polygon is less than the circumference of the circle,

because each side is less than the corresponding arc.

Hence the perimeter of the second polygon is *nearer* the circumference, but is less than the circumference.

By bisecting the arcs of the second polygon we should get a third polygon, whose perimeter is nearer the circumference of the circle than the second.

It is clear that by continuing this process, we can get a polygon whose perimeter is as near as we please to the circumference of the circle. Hence,

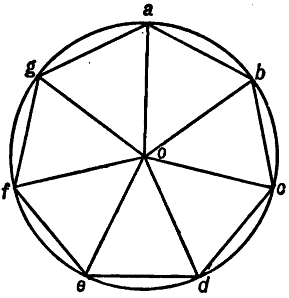
The length of the circumference of a circle is that to which the length of the perimeter of a regular inscribed polygon approaches, as the number of its sides is continually increased.

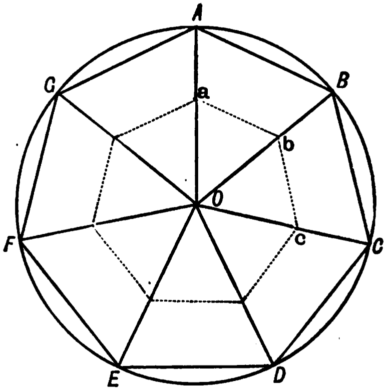
*33. Prop. *The ratio of the circumference of a circle to its diameter is the same for all circles.*

Let $ABCDEF$, $abcdef$ be any two circles.

Let a *regular polygon* of any, the same number of sides be inscribed in each of them.

Join A, B, C, etc., a, b, c, etc. the angular points of the polygons to the centres O, o respectively.





Then AOB any one of the isosceles triangles in the first figure is equiangular with*, and therefore similar to, aob any one of the isosceles triangles in the second figure.

$$\therefore\ AB : OA = ab : oa, \qquad \text{[Euc. VI. 4.]}$$

and

$$BC : OA = bc : oa,$$

and so on.

$$\therefore\ AB + BC + CD + \text{etc.} : OA = ab + bc + cd + \text{etc.} : oa;$$

or, the perimeter of the first polygon is to the radius of the first circle as the perimeter of the second polygon is to the radius of its circle.

---

* For at O and at o, four right angles are each divided into the *same* number of equal angles, so that the *vertical* angles AOB, aob of the isosceles triangles AOB, aob are equal.

This is true *whatever* be the number of the sides of the two polygons.

And therefore it is true *however great* be the number of sides of the two polygons.

But the *circumferences* of the circles are what the *perimeters* of the polygons become, when the number of the sides is indefinitely increased. Therefore the circumference of the first circle is to its radius OA as the circumference of the second circle is to its radius oa.

Thus the ratio of the circumference of any circle to its radius is equal to the ratio of the circumference of any other circle to its radius.

So that the ratio $\dfrac{\text{circumference}}{\text{diameter}}$ is the same numerical quantity for all circles. Q. E. D.

34. We said above (Art. 29) that this number is 3·14159 etc., and that it is denoted by π.

Hence the circumference of any circle of radius r

$$= (3\cdot14159265 \text{ etc.}) \times \text{its diameter} = 2\pi r.$$

35. We may notice that $\frac{22}{7} = 3\cdot\dot{1}4285\dot{7}$.

So that $\frac{22}{7}$ and π differ by less than a thousandth part of their value.

Also $\frac{355}{113} = 3\cdot1415929$ etc. So that $\frac{355}{113}$ may be used for π with sufficient accuracy for any practical purpose.

Hence $\frac{22}{7}$, 3·14159 and $\frac{355}{113}$ are each used for π according to the degree of accuracy required.

36. Of these 3·14159 is the most frequently used. The student should notice however that in *dividing* by π it will be more convenient to use $\frac{355}{113}$ than 3·14159.

37. The following results are instructive.

The ratio of the perimeter of a regular polygon inscribed in a circle to the diameter of the circle, when the polygon has

four sides, is $2\sqrt{2} = 2{\cdot}8284\ldots$

six sides, is $3 = 3$

eight sides, is $4\sqrt{(2-\sqrt{2})} = 3{\cdot}0614\ldots$

ten sides, is $\frac{5}{2}(\sqrt{5}-1) = 3{\cdot}0901\ldots$

twelve sides, is $3\sqrt{2}(\sqrt{3}-1) = 3{\cdot}1058\ldots$

twenty sides, is $5\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\} = 3{\cdot}1287\ldots$

sixty sides, is
$$\frac{15}{2\sqrt{2}}\{(\sqrt{5}-1)(\sqrt{3}+1) - \sqrt{(10+2\sqrt{5})}(\sqrt{3}-1)\} = 3{\cdot}1401\ldots$$

These numbers approach 3·14159 as the number of sides is increased, while the surd expression becomes more complicated.

The first of these results the student will be able to verify; the second is proved in Euclid IV. 15. The rest will be proved later on.

Example 1. The driving wheel of a locomotive engine is 5 ft. 6 in. high. What is its circumference?

Here we have a circle whose diameter is $5\frac{1}{2}$ feet;

$$\therefore \text{ its circumference} = \pi \times 5{\cdot}5 \text{ feet},$$
$$= (3{\cdot}14159\ldots) \times 5{\cdot}5 \text{ feet},$$
$$= 17{\cdot}278\ldots \text{ feet}.$$

The circumference is 17 ft. 3 in. approximately.

Example 2. If a piece of wire 1 foot long be bent into the form of a circle, what will be the diameter of the circle?

Here the circumference $= 1$ foot,

that is $\pi \times \text{diameter} = 1$ foot,

$$\therefore \text{ diameter} = \frac{1 \text{ foot}}{\pi} = \tfrac{7}{22} \times 1 \text{ foot}$$

$$= \tfrac{84}{22} \text{ inches}$$

$$= 3{\cdot}8 \text{ inches, nearly.}$$

If a greater degree of accuracy be desired, we must use $\frac{355}{113}$ instead of $\frac{22}{7}$.

We then get, the diameter $= 3{\cdot}7699$ inches,

$= 3{\cdot}77$ inches very nearly.

EXAMPLES. V.

In the answers of the first 12 of the following examples $\frac{22}{7}$ is used for π.

(1) Find the circumference of a circle whose diameter is one yard.

(2) Find the circumference of a circle whose radius is 4 feet.

(3) Find the circumference of a 48 inch bicycle wheel.

(4) The circumference of a circle is 10 feet; find its diameter.

(5) What must be the diameter of a locomotive driving wheel, that it may make 220 revolutions per mile?

(6) How many revolutions does a 36 inch bicycle wheel make per mile?

(7) How many more revolutions per mile does a 50 inch bicycle wheel make than one of 52 inches?

(8) A locomotive whose driving wheel is 5 feet high has an instrument to record the number of revolutions made. What number will the instrument record in running 100 miles?

(9) If the instrument in Question 8 indicates 3 revolutions per second, how many miles per hour is the engine running?

(10) What is the diameter of the driving wheel of a locomotive engine which makes 4 revolutions per second when the engine is going at the rate of 60 miles per hour?

(11) The large hand of the Westminster clock is 11 feet long; how many yards per day does its extremity travel? How far does the extremity move in a minute?

(12) The diameter of the whispering gallery in St Paul's is 108 feet; what is its circumference?

(13) Find the number of inches of wire necessary to construct a figure consisting of a circle with a regular hexagon inscribed in it, one of whose sides is 3 feet.

(14) How many inches of wire would be necessary in a figure similar to that in Question (13), if the circumference of the circle were ten feet?

(15) Find how many inches of wire are necessary to make a figure consisting of a circle and a square inscribed in it, when each side of the square is 2 feet.

(16) How many inches of wire are necessary for a figure similar to that in Question (15), when the circumference of the circle is 12 feet?

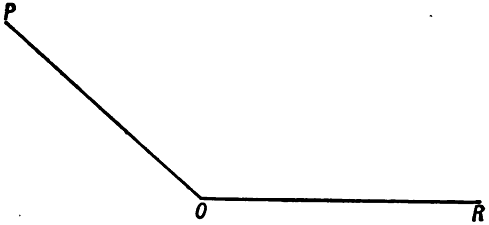
(17) Find the length of string necessary to string the handle of a cricket bat; having given the diameter of the handle $= 1\frac{1}{4}$ in., the length of the handle $= 12$ in., the diameter of the string $= \frac{1}{40}$th of an inch.

CHAPTER IV.

ON THE MEASUREMENT OF ANGLES.

38. IN elementary Geometry (Euclid I.—VI.) the angles considered are each always less than two right angles.

For example, in speaking of the angle *ROP* in Euclid we should always mean the angle less than two right angles,



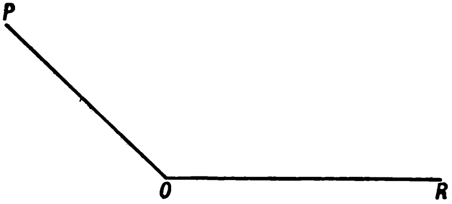
not an angle measured in the opposite direction greater than two right angles.

39. In Trigonometry, by *the angle ROP* is meant, not the *present inclination* of the two lines *OR*, *OP* but, the **amount of turning** which *OP* has gone through when, starting from the position *OR*, it has turned about *O* into the position *OP*.

Example. Suppose a race run round a circular course. The position of any one of the competitors would be known, if we remark that he has described a certain angle about the centre of the course. Thus, if the distance to be run is three times round, the line joining each competitor to the centre would have to describe an angle of 12 right angles.

When we remark that a competitor has described an angle of $6\frac{2}{3}$ right angles, we record not only his present position, but the total distance he has gone. He would in such a case have gone a little more than one and a half times round the course.

40. DEFINITION. The **angle** between two lines OR, OP is the **amount of turning** about the point O which one of



the lines OP has gone through in turning from the position OR into the position OP.

41. The angle ROP may be the *geometrical* representative of an unlimited number of Trigonometrical angles.

(i) The angle ROP may represent the angle less than two right angles as in Euclid.

In this case OP has turned from the position OR into the position OP by turning about O in the direction *contrary* to that of the hands of a watch.

(ii) The angle ROP may represent the angle described by OP in turning from the position OR into the position OP in the *same* direction as the hands of a watch.

In the first case it is usual to say that the angle ROP is described in the *positive* direction, in the second that the angle is described in the *negative* direction.

(iii) The angle ROP may be the geometrical representation of any of the Trigonometrical angles formed by any number of complete revolutions in the *positive* or in the *negative* direction, added to either of the first two angles. (We shall return to this subject in Chapter IX.)

EXAMPLES. VI.

Give a geometrical representation of each of the following angles, the starting line being drawn in each case from the turning point towards the right.

1. $+3$ right angles.
2. $+5$ right angles.
3. $+4\frac{1}{2}$ right angles.
4. $+7\frac{1}{4}$ right angles.
5. -1 right angle.
6. $10\frac{2}{3}$ right angles.
7. $-10\frac{1}{3}$ right angles.
8. $+4$ right angles.
9. -4 right angles.
10. $4n$ right angles.
11. $(4n+2)$ right angles.
12. $-(4n+\frac{1}{2})$ right angles.

---

42. There are two methods of measuring angles.

(i) The rectangular measure.

(ii) The circular measure.

RECTANGULAR MEASURE.

43. Angles are always measured *in practice* with the **right angle** (or part of the right angle) as unit.

44. The reasons why the right angle is chosen for a unit are:

(i) All right angles are equal to one another.

(ii) A right angle is practically easy to draw.

(iii) It is an angle whose size is very familiar.

45. The right angle is a large angle, and it is therefore subdivided for practical purposes.

It is usual to explain two methods of subdivision.

(i) The sexagesimal method.

(ii) The centesimal, or decimal, method *.

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* See Art. 55.

I. *The Sexagesimal Method.*

46. In this method the right angle is divided into 90 equal parts, each of which is called **a degree**; each degree is subdivided into 60 equal parts, each of which is called **a minute**; and each minute is again subdivided into 60 equal parts, each of which is called a **second**.

Instruments used for measuring angles are subdivided accordingly; and the size of an angle is known when, with such an instrument, it has been observed that the angle contains a certain number of degrees, and a certain number of minutes beyond the number of complete degrees, and a certain number of seconds beyond the number of complete minutes.

Thus an angle might be recorded as containing 79 degrees + 18 minutes + 36·4 seconds.

Degrees, minutes, and seconds are indicated respectively by the symbols °, ′, ″, and the above angle would be written

$$79^\circ\,.\,18'\,.\,36{\cdot}4''.$$

* II. *The Centesimal or Decimal Method.*

47. The other method of subdivision is the Centesimal or Decimal. Here each right angle is divided into 100 equal parts each of which is called a **grade**; each grade is subdivided into 100 equal parts, each of which is called a **minute**; and each minute is again subdivided into 100 equal parts, each of which is called a **second**.

Instruments of observation would be subdivided accordingly, and any observed angle would be recorded as containing so many grades + so many minutes + so many seconds.

Grades, minutes and seconds are indicated respectively by the symbols ^{g}, $^{\backprime}$, $^{\backprime\backprime}$. So that an angle of 26 grades + 19 minutes + 34·2 seconds would be written

$$36^{g}\ 19^{\backprime}\ 34{\cdot}2^{\backprime\backprime}.$$

*48. It will be observed that this method is simply that of the decimal system of notation.

The above angle for example $=\frac{36}{100}+\frac{19}{10000}+\frac{34\cdot2}{1000000}$ of a right angle.

That is ·3619342 of a right angle.
This is equal to 36·19342 of a grade.
Also to 3619·342 of a minute.
Also to 361934·2 of a second.

Example. Express $302^{g}\ 2^{\backprime}\ 4{\cdot}6^{\backprime\backprime}$ as the decimal of a right angle.

$$\text{This angle} = \frac{302}{100}+\frac{2}{10000}+\frac{4\cdot6}{1000000} \text{ of a right angle}$$
$$= 3{\cdot}0202046 \text{ of a right angle.}$$

*49. Hence, to express an angle given in grades, minutes and seconds as the decimal of a right angle, we have only to observe that the *first and second* decimal places are occupied by the grades, the *third and fourth* decimal places are occupied by the minutes, and the *fifth and sixth* decimal places are occupied by the seconds.

*50. The same observation will enable us to express in grades, minutes, and seconds an angle given as the decimal of a right angle.

Example. Express 3·4650023 of a right angle in grades, minutes, and seconds.

3 right angles = 300 grades.
·46 of a right angle = 46 grades.
·00,50 of a right angle = 50 minutes.
·00,00,02,3 of a right angle = 2·3 seconds.

Therefore the angle is $346^{g}\ 50^{\backprime}\ 2{\cdot}3^{\backprime\backprime}$.

*EXAMPLES. VII.

Express as the decimal of a right angle,

(1) $63^{g}\ 21^{\backprime}\ 18^{\backprime\backprime}$.

(2) $104^{g}\ 26^{\backprime}\ 99{\cdot}1^{\backprime\backprime}$.

(3) $2^{g}\ 18^{\backprime}\ 27^{\backprime\backprime}$.

(4) $3^{g}\ 29^{\backprime}\ 48{\cdot}94^{\backprime\backprime}$.

(5) $62^{\backprime}\ 41^{\backprime\backprime}$.

(6) $1000^{g}\ 8^{\backprime}\ 12^{\backprime\backprime}$.

(7) $32^{g}\ 4^{\backprime}\ 5{\cdot}2^{\backprime}$

(8) $1^{g}\ 2^{\backprime}\ 3{\cdot}4^{\backprime\backprime}$

(9) $69^{g}\ 0^{\backprime}\ 7{\cdot}1^{\backprime\backprime}$.

(10) $119^{g}\ 3^{\backprime}\ 0{\cdot}45^{\backprime\backprime}$.

(11) $1006^{g}\ 18^{\backprime}\ 1^{\backprime\backprime}$

(12) $2^{g}\ 26^{\backprime}\ 4{\cdot}8^{\backprime}$.

Express in grades, minutes and seconds,

(13) ·367891 of a right angle.

(14) 1·043021 of a right angle.

(15) ·012003 of a right angle.

(16) ·00102 of a right angle.

(17) ·0625 of a right angle.

(18) 3·02125 of a right angle.

(19) 1·001 of a right angle.

(20) ·0101001 of a right angle.

(21) 6·451 of a right angle.

(22) ·023 of a right angle.

(23) ·00011 of a right angle.

(24) ·00001 of a right angle.

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51. An angle given in degrees, minutes, and seconds may be expressed as the decimal of a right angle by the usual method.

Example. Express $39^{0}\ 4'\ 27''$ as the decimal of a right angle.

60) 27 seconds
60) 4·45 minutes
90) 39·07416666 etc. degrees
·43415740740 etc. right angles

Answer. $\cdot 43415\dot{7}4\dot{0}$ of a right angle.

52. An angle given as the decimal of a right angle may be expressed in degrees, minutes, and seconds by the converse of the above.

Example. Express $\cdot 43415\dot{7}4\dot{0}$ of a right angle in degrees, minutes, and seconds.

$$\begin{array}{r l} \cdot 43415740740 & \text{etc. right angles} \\ 90 & \\ \hline 39\cdot 07416666600 & \text{degrees} \end{array}$$

The last two figures would be 66 if we were to write down the recurring part to more figures.

$$\begin{array}{l r l} \text{This gives} & 39\cdot 07416666666 & \text{etc. degrees} \\ & 60 & \\ \hline & 4\cdot 4499999960 & \text{minutes} \\ \text{that is} & 4\cdot 44\dot{9} & \text{minutes} \\ \text{or} & 4\cdot 45 & \text{minutes} \\ & 60 & \\ \hline & 27\cdot 00 & \text{seconds.} \end{array}$$

The result is $39^{0}\ 4'\ 27''$.

*53. We have seen that an angle expressed as the decimal of a right angle can be at once expressed in grades, minutes, and seconds.

Hence an angle expressed in degrees, minutes, and seconds, can be expressed in grades, etc. by first reducing the angle to the decimal of a right angle.

Example. Express $39^{0}\ 4'\ 27''$ in grades, minutes, and seconds.

This angle is $\cdot 43415\dot{7}4\dot{0}$ of a right angle, by Art. 51
and this $= 43^{g}\ 41^{`}\ 57\cdot\dot{4}0\dot{7}^{``}$.

*54. An angle given in grades, minutes, and seconds can be expressed in degrees, minutes, and seconds by first expressing the angle as the decimal of a right angle.

Example. Express $\cdot 43^{g}\ 41^{\backprime}\ 57\cdot 407^{\backprime\backprime}$ in degrees, minutes, and seconds.

This angle is $\cdot 43415\dot{7}4\dot{0}$ of a right angle, which is $39^{0}\ 4'\ 27''$ from Art. 52.

* EXAMPLES. VIII.

Express each of the following angles (i) as the decimal of a right angle, (ii) in grades, minutes, and seconds;

(1) $8^{0}\ 15'\ 27''$.

(2) $6^{0}\ 4'\ 30''$.

(3) $97^{0}\ 5'\ 15''$.

(4) $16^{0}\ 14'\ 19''$.

(5) $132^{0}\ 6'$.

(6) 49^{0}.

Express in degrees, minutes and seconds,

(7) $1^{g}\ 37^{\backprime}\ 50^{\backprime\backprime}$.

(8) $8^{g}\ 75^{\backprime}$.

(9) $170^{g}\ 45^{\backprime}\ 35^{\backprime\backprime}$.

(10) $24^{g}\ 0^{\backprime}\ 25^{\backprime\backprime}$.

(11) $18^{g}\ 1^{\backprime}\ 15^{\backprime\backprime}$.

(12) 35^{g}.

55. The decimal or centesimal system of subdividing a right angle was proposed by the French at the commencement of the present century; but, although it possesses many advantages over the established method, no one has been found willing to undertake the great expense that would have to be incurred in rearranging all tables and all books of reference, and all the records of observations, which would have to be transferred from the old system to the new, before the advantages of the decimal system could be felt. Thus the decimal system of angular measurement has never been used even in France, and in all probability never will be used in practical work.

The [illegible] of Paris [illegible]

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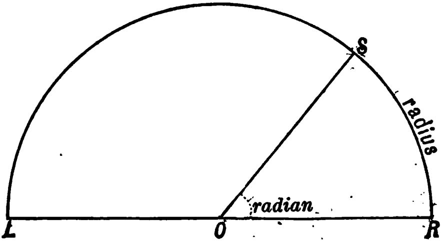
On Circular Measure.

56. By the following construction we get an angle of great importance in Trigonometry.

On the circumference of a circle whose centre is *O*



let an *arc RS* be measured so that its length is equal to the *radius* of the circle, and let *R* and *S* be joined to the centre.

57. We are about to prove (Art. 60) that this angle *ROS* is a *fixed* fraction of a right angle, so that all such angles are equal to one another.

58. We may state the same thing thus—We are about to prove that if we take any number of different circles, and measure on the circumference of each an arc equal in length to its radius, then the angles at the centres of these circles which stand on these arcs respectively, will be all of the same size.

59. Definition. The angle which at the centre of a circle stands on an arc equal in length to the radius of the circle is called a **Radian.**

60. *To prove that all Radians are equal to one another.*

Since the Radian at the centre of a circle stands on an arc equal in length to the radius,

and an angle of two right angles at the centre of a circle stands on half the circumference, and since angles at the centre of a circle are to one another as the arcs on which they stand (Euc. VI. 33),

therefore the radian is to an angle of two right angles as the radius is to half the circumference;
that is, as the diameter is to the whole circumference;
that is, in the constant ratio $1 : \pi$.

Therefore the radian $= \dfrac{2 \text{ right angles}}{\pi}$.

That is, the radian is a fixed fraction of a right angle.
But all right angles are equal to one another.
Therefore all radians are equal to one another. Q. E. D.

61. Thus the radian possesses the qualification most essential in a unit, viz. it is always the same.

The student will find, in the theoretical part of Trigonometry, that many expressions can be written more shortly when a radian is used for the unit of angle, than when any other unit is used.

62. Thus the reasons why a radian is used as a unit are:

(i) All radians are equal to one another.

(ii) Its use simplifies many formulæ in Theoretical Trigonometry.

63. The system of angular measurement in which a radian is the unit is called **Circular Measure.**

Therefore the *circular measure of an angle* is the *number of radians* which the angle contains.

64. A radian $= \dfrac{1}{\pi} \times 2$ right angles,

$= \frac{113}{355}$ of 180° nearly,

$= 57{\cdot}2957 \ldots\ldots$ degrees.

The student should notice that a radian is a little less than an angle of an equilateral triangle.

65. Circular measure is, as we have said, used in *theoretical* investigations, in which the angle under consideration is almost always expressed by a *letter*. This is usually one of the *Greek* letters $\alpha, \beta, \gamma \ldots, \phi, \theta, \psi \ldots$.

Strictly speaking these letters represent *numbers*, i.e. measures; so that some unit of *angle* must be *understood* in such an expression as 'the angle θ.' (Art. 2.)

For this reason, when an angle is denoted by a *Greek* letter such as α, β, γ, etc., θ, ϕ, ψ, etc., it is understood that *circular measure* is the measure used, unless the contrary is *expressly stated.*

So that 'the angle θ' means 'θ radians.'

Similarly 'the angle π' means 'π radians' or '3·14159... radians,' that is two right angles.

[*Note.* It will also be convenient, in using such letters as A, B, $C \ldots S$, T, etc. to represent angles, to agree that the *unit* understood with this kind of letter shall be a *degree*, so that when A stands for an angle, that angle contains A degrees.]

66. In *numerical* examples it will be necessary to use some letter (c suppose) to denote a radian.

Thus 2^c denotes 'two radians.'

67. Strictly speaking then 'the angle θ' should be written θ^c. (Just as in speaking of 'the angle ninety,' we ought to say ninety *degrees.*) But if it is clearly understood that 'the angle θ' means θ *radians*, there can be no ambiguity in the expression.

68. The student cannot too carefully notice, that unless an *angle* is obviously referred to, the letters $\theta, \phi, \ldots \alpha, \beta, \ldots$ stand for *mere numbers.*

Thus as we have said above (29) π *stands for a number and a number only,* viz. 3·14159......, but in the expression 'the angle π' that is 'the angle 3·14159......' there must be *some unit* understood. The *unit understood* here is a radian, and therefore 'the angle π' stands for 3·14159......^{c}, that is *two right angles.*

Hence, *when an angle is referred to, π is a very convenient abbreviation for* **two right angles.**

69. To express in degrees or grades an angle given in radians, we first express the angle in right angles, remembering that

$$\textbf{2 right angles} = \pi \textbf{ radians.}$$

Example. How many degrees are there in the angle whose circular measure is 2?

$$\text{This angle} = 2 \text{ radians } = 2 \times \frac{2 \text{ right angles}}{\pi} = \frac{4}{\pi} \text{ right angles,}$$

$$= \frac{4 \times 90^0}{\pi} = \frac{360^0}{\pi},$$

$$\therefore \text{ the angle contains } \frac{360}{\pi} \text{ degrees.}$$

*70. If D, G and a be the number of degrees, grades and radians respectively in any angle, then

$$\frac{D}{180} = \frac{G}{200} = \frac{a}{\pi}.$$

For each fraction is the ratio of the angle to two right angles.

Example. Find the number of degrees in two radians.

Let D be the number, then

$$\frac{D}{180}=\frac{2}{\pi},$$

$$\therefore\ D=\frac{360}{\pi}.$$

EXAMPLES. IX.

1. Express the following angles in rectangular measure.

(1) π. (2) $\frac{3\pi}{4}$. (3) 1^c.

(4) 3^c. (5) $3\cdot14159265^c$ etc. (6) $\frac{2^c}{\pi}$.

(7) θ. (8) $\cdot00314159^c$ etc. (9) 10π.

2. Express the following angles in circular measure.

(1) 180^0. (2) 360^0. (3) 60^0.

(4) $22\frac{1}{2}^0$. (5) 1^0. (6) $57\cdot295^0$ etc.

(7) n^0. (8) $\frac{90^0}{\pi}$. (9) A.

*3. Express the following angles in circular measure.

(1) $33^g\ 33^{\backprime}\ 33\cdot\dot{3}^{\backprime\backprime}$. (2) 50^g. (3) $16\cdot\dot{6}^g$.

(4) 1^g. (5) $1^{\backprime}$. (6) $10^{\backprime\backprime}$.

(7) n^g. (8) $\frac{200^g}{\pi}$. (9) 1000^g.

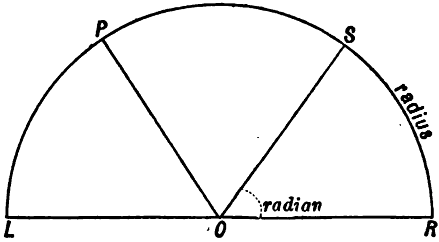
*4. Find the ratio of

(1) 45^0 to $\frac{3\pi}{4}$. (2) 60^0 to 60^g.

(3) 25^g to $22^0\,30'$. (4) 24^g to 2^c.

(5) $1\cdot75^c$ to $\frac{100^0}{\pi}$. (6) 1^0 to 1^c.

*71. *To prove that the measure of an angle at the centre of a circle in radians* (i.e. *in Circular Measure*) *is the ratio of the arc on which it stands to the radius of the circle.*



Since angles at the centre of a circle are to one another as the arcs on which they stand (Euc. VI. 33),

Therefore any angle ROP at the centre of a circle is to the radian as its arc RP is to the arc of the radian.

But the arc of the radian is equal to the radius,

Therefore $\dfrac{\text{any angle } ROP}{\text{the radian}}$ is equal to $\dfrac{\text{its arc } RP}{\text{the radius}}$.

And therefore any angle ROP is equal to $\dfrac{\text{its arc}}{\text{radius}} \times$ (a radian).

That is, the *measure* a of an angle *in radians* is the ratio $\dfrac{\text{arc}}{\text{radius}}$; or, $a = \dfrac{\text{arc}}{\text{radius}}$.

Hence (cf. Art. 70) $\dfrac{D}{180} = \dfrac{G}{200} = \dfrac{a}{\pi} = \dfrac{\text{arc}}{\text{radius}} \times \dfrac{1}{\pi}$.

Example. Find the number of grades in the angle subtended by an arc 46 ft. 9 in. long, at the centre of a circle whose radius is 25 feet.

The angle stands on an arc of $46\frac{3}{4}$ ft. and the radian, at the centre of the same circle, stands on an arc of 25 feet.

$$\therefore \text{ the angle} = \frac{46\frac{3}{4}}{25} \text{ radians}, \quad = \tfrac{187}{100} \times \frac{\text{2 right angles}}{\pi},$$

$$= \tfrac{187}{100} \times \frac{200^g}{\pi}, = 119^g \text{ nearly.}$$

* EXAMPLES. X.

(In the answers $\frac{22}{7}$ is used for π.)

(1) Find the number of radians in an angle at the centre of a circle of radius 25 feet, which stands on an arc of $37\frac{1}{2}$ feet.

(2) Find the number of degrees in an angle at the centre of a circle of radius 10 feet, which stands on an arc of 5π feet.

(3) Find the number of right angles in the angle at the centre of a circle of radius $3\frac{2}{11}$ inches, which stands on an arc of 2 feet.

(4) Find the number of French minutes in the angle at the centre of a circle of radius 8 ft. 4 inches, which stands on an arc of 1 inch.

(5) Find the length of the arc subtending an angle of $4\frac{1}{2}$ radians at the centre of a circle whose radius is 25 feet.

(6) Find the length of an arc of eighty degrees on a circle of 4 feet radius.

(7) Find the length of an arc of sixty grades on a circle of ten feet radius.

(8) The angle subtended by the diameter of the Sun at the eye of an observer is $32'$; find approximately the diameter of the Sun if its distance from the observer be 90,000,000 miles.

(9) A railway train is travelling on a curve of half a mile radius at the rate of 20 miles an hour; through what angle has it turned in 10 seconds?

(10) A railway train is travelling on a curve of two-thirds of a mile radius, at the rate of 60 miles an hour; through what angle has it turned in a quarter of a minute?

(11) Find approximately the number of English seconds contained in the angle which subtends an arc one mile in length at the centre of a circle whose radius is 4000 miles.

(12) If the radius of a circle be 4000 miles, find the length of an arc which subtends an angle of $1''$ at the centre of the circle.

(13) If in a circle whose radius is 12 ft. 6 in. an arc whose length is ·6545 of a foot subtends an angle of 3 degrees, what is the ratio of the diameter of a circle to its circumference?

(14) If an arc 1·309 feet long subtend an angle of $7\frac{1}{2}$ degrees at the centre of a circle whose radius is 10 feet, find the ratio of the circumference of a circle to its diameter.

(15) On a circle 80 feet in radius it was found that an angle of $22^0\ 30'$ at the centre was subtended by an arc 31 ft. 5 in. in length; hence calculate to four decimal places the numerical value of the ratio of the circumference of a circle to its diameter.

(16) If the diameter of the moon subtend an angle of $30'$, at the eye of an observer, and the diameter of the sun an angle of $32'$, and if the distance of the sun be 375 times the distance of the moon, find the ratio of the diameter of the sun to that of the moon.

(17) Find the number of radians in (i.e. the circular measure of) $10''$ correct to 3 significant figures. (Use $\frac{355}{113}$ for π.)

(18) Find the radius of a globe such that the distance measured upon its surface between two places in the same meridian, whose latitudes differ by $1^0\ 10'$, may be one inch.

(19) Two circles touch the base of an isosceles triangle at its middle point, one having its centre at, and the other passing through the vertex. If the arc of the greater circle included within the triangle be equal to the arc of the lesser circle without the triangle, find the vertical angle of the triangle.

(20) By the construction in Euc. I. 1, prove that the unit of circular measure is less than 60^0.

(21) On the 31st December the Sun subtends an angle of 32′ 36″, and on 1st July an angle of 31′ 32″; find the ratio of the distances of the Sun from the observer on those two days.

(22) Show that the measure of the angle at the centre of a circle of radius r, which stands on an arc a, is $\frac{k\,.\,a}{r}$, where k depends solely on the unit of angle employed.

Find k when the unit is (i) a radian, (ii) a degree.

*72. Questions concerning angles expressed in different systems of measurement are easily solved by *expressing each angle in right angles.*

Example 1. The sum of the measure of an angle in degrees and twice its measure in radians is $23\frac{2}{7}$, find its measure in degrees $(\pi=\frac{22}{7})$.

Let the angle contain x right angles.

Then the measure of the angle in degrees $=90x$,

„ „ „ „ „ radians $=\frac{\pi}{2}x$.

$$\therefore\ 90x+2\,.\,\frac{\pi}{2}x=23\tfrac{2}{7},$$

$$\therefore\ 90x+\tfrac{22}{7}x=\tfrac{163}{7},$$

$$\therefore\ 652x=163,$$

$$\therefore\ x=\tfrac{1}{4}.$$

The angle is $\frac{1}{4}$ of a right angle, that is $22\frac{1}{2}°$, nearly.

Example 2. The three angles of a triangle are in arithmetical progression, and the measure of the least in grades is to that of the greatest in circular measure as $120 : \pi$. Express each angle in degrees.

Let the angles contain $x-y$, x, $x+y$ **right angles** respectively; they are then in A.P.

Their *sum* is $3x$ right angles; but since they are the angles of a triangle, their *sum* is 2 right angles;

$$\therefore 3x=2,$$

$$\therefore x=\tfrac{2}{3}.$$

Again, the least angle contains $(x-y)\times 100$ grades, and the greatest angle contains $(x+y)\dfrac{\pi}{2}$ radians,

$$\therefore 100\,(x-y) : \frac{\pi}{2}(x+y)=120 : \pi.$$

$$\therefore 100\,(x-y)=60\,(x+y),$$

$$\text{or, } 40x=160y,$$

$$\text{or, } x=4y.$$

$$\therefore 4y=\tfrac{2}{3},$$

$$\text{because } x=\tfrac{2}{3},$$

$$\text{or, } y=\tfrac{1}{6}.$$

$$\therefore x-y=\tfrac{2}{3}-\tfrac{1}{6}=\tfrac{1}{2},\quad x=\tfrac{2}{3},\quad x+y=\tfrac{2}{3}+\tfrac{1}{6}=\tfrac{5}{6}.$$

Thus the angles contain $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{5}{6}$ right angles respectively;

therefore the angles are 45^0, 60^0, 75^0.

* * EXAMPLES. XI.

(In the following examples the answers will be given in terms of π.)

(1) The sum of the degrees and of the grades in a certain angle is 38; find its circular measure.

(2) The difference of two angles is 20^g, and their sum is 48^0; find them.

(3) One angle is double of a second, and the sum of their measures in degrees and in grades respectively is 140; express the angles in degrees.

(4) Two angles are in the ratio of 4 : 5, and the difference of their measures in grades and in degrees respectively is $2\frac{1}{2}$; find the angles in degrees.

(5) The difference between two angles is $\frac{\pi}{9}$, and their sum is 56 degrees; find the angles.

(6) If the three angles of a triangle are in arithmetical progression, show that the mean angle is 60^0.

(7) The three angles of a triangle are in arithmetical progression, and the number of grades in the least is to the number of degrees in the mean as 5 : 6. Find the angles in degrees.

(8) The three angles of a triangle are in arithmetical progression, and the number of grades in the greatest is to the number of degrees in the sum of the other two as 10 : 11. Find the angles in degrees.

(9) The three angles of a triangle are in arithmetical progression, and the number of grades in the least is to the number of radians in the greatest as $200 : 3\pi$. Express the angles in grades.

(10) If D be the number of degrees and G the number of grades in any angle, prove that $G-D=\frac{1}{9}D$.

(11) If M be the number of English minutes and m the number of French minutes in any angle, prove that

$$2M-m=\tfrac{4}{27}M.$$

(12) If G, D and C be the number of grades, degrees and radians in any angle, prove that $G-D=\frac{20C}{\pi}$.

(13) If an angle be expressed in French minutes, show that it will be transferred to English minutes by multiplying by ·54.

(14) Divide $33^\circ\ 6'$ into two parts so that the number of English seconds in one part may be equal to the number of French seconds in the other part.

(15) Find the ratio of $9^\circ\ 27'$ to $12^g\ 50^\backprime$.

(16) Find the number of radians in an angle of n English minutes.

(17) Express in each of the three systems of angular measurement the angles

- (i) of a regular hexagon,
- (ii) of a regular octagon,
- (iii) of a regular quindecagon.

(18) Show that the number of degrees in an angle of a regular decagon is to the number of grades in an angle of a regular pentagon in the ratio of 6 : 5.

(19) Show that the number of grades in an angle of a regular pentagon is equal to the number of degrees in an angle of a regular hexagon.

(20) Find in English minutes the difference between the angle of a regular polygon of 48 sides and two right angles.

(21) If we take for unit the angle between a side of a regular quindecagon and the next side produced, find the measures (i) of a right angle, (ii) of a radian.

(22) Find the unit when the sum of the measures of a degree and of a grade is 1.

(23) What is the unit when the sum of the measures of 9° and of 5^g is $\frac{3}{20}$?

(24) If the measure of b grades is a, find the measure of c degrees.

(25) What is the unit when the sum of the measures of a grades and of b degrees is c?

(26) The number of grades in a certain angle exceeds the number of degrees in it by $\frac{1}{10}$ of the number of degrees in a radian. If this angle be taken as unit, what is the measure of a right angle?

(27) The three numbers which express the three angles of a triangle are all equal, and the units of angle in each are respectively a degree, a grade and the sum of a degree and a grade; express each of the angles in circular measure.

(28) The three angles of a triangle have the same measure when expressed in degrees, grades and radians respectively; find this measure.

(29) The measures of the angles of a triangle in degrees, grades and radians respectively are in the ratio of 1 : 10 : 100; find the number of radians in the smallest angle.

(30) The interior angles of an irregular polygon are in A. P.; the least angle is 120^0; and the common difference 5^0: find the number of sides.

CHAPTER V.

THE TRIGONOMETRICAL RATIOS.

73. Let *ROE* be any angle (see the figure in Art. 83). In one of the lines containing the angle take any point *P*, and from *P* draw *PM* perpendicular to the other line *OR*.

Then, in the right-angled triangle *OPM*, formed from the angle *ROE*,

(i) the side *MP*, which is *opposite the angle under consideration*, is called the **perpendicular**;

(ii) the side *OP*, which is *opposite the right angle*, is called the **hypotenuse**;

(iii) the *third* side *OM*, which is adjacent to the right angle and to the angle under consideration, is called the **base.**

From these three,—perpendicular, hypotenuse, base,—we can form *three* different sets containing two each.

The **ratios** or *fractions* formed from these sets, viz.

$$\text{(i) } \frac{\text{perpendicular}}{\text{hypotenuse}}, \quad \text{(ii) } \frac{\text{base}}{\text{hypotenuse}}, \quad \text{(iii) } \frac{\text{perpendicular}}{\text{base}},$$

and the ratios formed by inverting each of them, viz.

$$\text{(iv) } \frac{\text{hypotenuse}}{\text{perpendicular}}, \quad \text{(v) } \frac{\text{hypotenuse}}{\text{base}}, \quad \text{(vi) } \frac{\text{base}}{\text{perpendicular}},$$

will be found to be of great importance in treating of any angle *ROE*. Accordingly to each of these six ratios has been given a separate *name* (Art. 75).

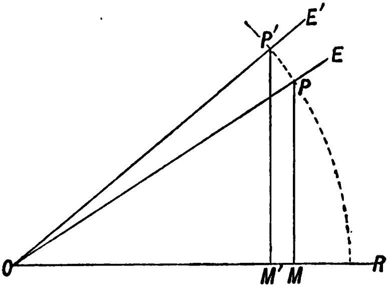
74. The student should observe carefully

(i) that each *ratio*, such as $\frac{\text{perpendicular}}{\text{hypotenuse}}$, is a mere *number;*

(ii) that, as we shall prove in Art. 83, these ratios remain unchanged as long as the angle remains unchanged;

(iii) that if the angle be altered ever so slightly, there is a consequent alteration in the value of these ratios.

[For, let ROE, ROE' be two angles which are nearly equal;

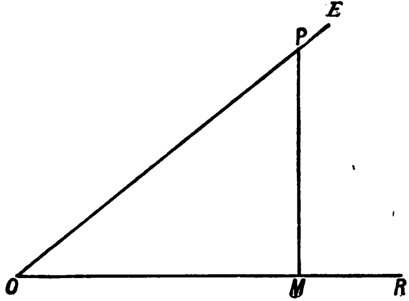


Let $OP=OP'$; then OM is *not*$=OM'$, and therefore the ratios $\frac{OM}{OP}$ and $\frac{OM'}{OP'}$ are *not* equal; also MP is *not*$=M'P'$ and therefore the ratios $\frac{MP}{OP}$ and $\frac{M'P'}{OP'}$ are *not* equal.]

(iv) that by giving **names** to these ratios we are enabled to apply the methods of Algebra to the Geometry of Euclid VI., just as in Chapter I. we applied the methods of Algebra to Euc. I. 47.

The student is recommended to pay careful attention to the following definitions. He should be able to write them out in the exact words in which they are printed.

75. **Definition.** *To define the three principal Trigonometrical Ratios of an angle.*



Let *ROE* be an angle.

In *OE* one of the lines containing the angle take any point *P*, and from *P* draw *PM* perpendicular to the other line *OR*, or, if necessary, to *RO* produced.

Then, in the right-angled triangle *OPM*, the side *MP*, which is *opposite the angle under consideration*, is called the *perpendicular*.

The side *OP*, which is *opposite the right angle*, is called the *hypotenuse*.

The *third* side *OM* (which is adjacent to the right angle and to the angle under consideration) is called the *base*.

Then the ratio

(i) $\frac{MP}{OP} = \frac{\text{perpendicular}}{\text{hypotenuse}}$ is called the **sine** of the angle *ROE*.

(ii) $\frac{OM}{OP} = \frac{\text{base}}{\text{hypotenuse}}$ " **cosine** "

(iii) $\frac{MP}{OM} = \frac{\text{perpendicular}}{\text{base}}$ " **tangent** "

These three are the principal Trigonometrical Ratios of the angle *ROE*.

76. If A stand for the angle ROE, these ratios are called sine A, cosine A and tangent A, and are usually abbreviated thus:

$$\sin A, \qquad \cos A, \qquad \tan A.$$

77. There are three other Trigonometrical Ratios, formed by *inverting* the sine, cosine and tangent respectively, which are called the cosecant, secant, and cotangent respectively.

78. *To define the three other Trigonometrical Ratios of any angle.*

The same construction and figure as in Art. 75 being made, then the ratio

(iv) $\dfrac{OP}{MP} = \dfrac{\text{hypotenuse}}{\text{perpendicular}}$ is called the **cosecant** of the angle ROE.

(v) $\dfrac{OP}{OM} = \dfrac{\text{hypotenuse}}{\text{base}}$ „ **secant** „

(vi) $\dfrac{OM}{MP} = \dfrac{\text{base}}{\text{perpendicular}}$ „ **cotangent** „

79. Thus if A stand as before for the angle ROE, these ratios are called cosecant A, secant A, and cotangent A. They are abbreviated thus,

$$\operatorname{cosec} A, \qquad \sec A, \qquad \cot A.$$

80. From the definition it is clear that

$$\operatorname{cosec} A = \frac{1}{\sin A},$$

$$\sec A = \frac{1}{\cos A},$$

$$\cot A = \frac{1}{\tan A}.$$

81. The above definitions **apply to an angle of any magnitude.** (We shall return to this subject in Chapter X.)

For the present the student may confine his attention to angles which are each less than a right angle.

82. The powers of the Trigonometrical Ratios are expressed as follows:

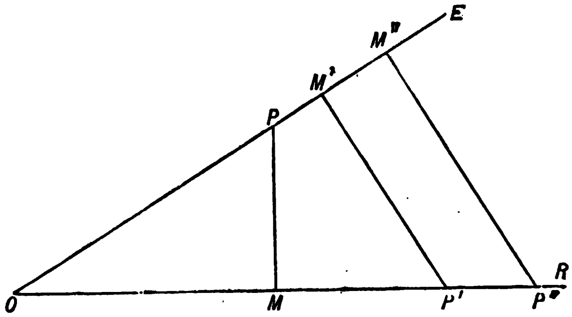
$$(\sin A)^2, \text{ i.e. } \left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2, \text{ is written } \quad \sin^2 A,$$

$$(\cos A)^2, \text{ i.e. } \left(\frac{\text{base}}{\text{hypotenuse}}\right)^2, \text{ is written } \quad \cos^2 A,$$

and so on.

The student must notice that '$\sin A$' is a *single symbol.* It is the *name* of a *number*, or *fraction*, belonging to the angle A; and if it be at any time convenient, we may denote *sin* A by a *single letter*, such as s or x. Also $\sin^2 A$ is an abbreviation for $(\sin A)^2$, that is for $(\sin A) \times (\sin A)$. Such abbreviations are used because they are *convenient.*

83. *The Trigonometrical Ratios are always the same for the same angle.*



Take any angle ROE; let P be any point in OE one of the lines containing the angle, and let P', P'' be any two points in OR the other line containing the angle. Draw PM perpendicular to OR, and $P'M'$, $P''M''$ perpendiculars to OE.

Then the three triangles OMP, $OM'P'$, $OM''P''$ each contain a right angle, and they have the angle at O common; therefore their third angles must be equal.

Thus the three triangles are equiangular.

Therefore the ratios $\frac{MP}{OP}$, $\frac{M'P'}{OP'}$, $\frac{M''P''}{OP''}$ are all equal. (Eu. VI. 4.)

But each of these ratios is $\frac{\text{perpendicular}}{\text{hypotenuse}}$ *with reference to the angle at* O; that is, they are each *sin* ROE.

Thus, sin ROE is the same *whatever* be the position of the point P on *either* of the lines containing the angle ROE.

Therefore sin ROE is always the same.

84. A similar proof holds good for each of the other ratios.

85. Also if two angles are equal, it is clear that the numerical values of their Trigonometrical Ratios will be the same.

We have already shown (Art. 74), that the values of these ratios are different for different angles.

Hence for each particular value of A, sin A, cos A, tan A, etc. have *definite numerical values.*

Example. We shall prove (Art. 92) that

$$\sin 30^\circ = \tfrac{1}{2} = \cdot 5, \quad \cos 30^\circ = \frac{\sqrt{3}}{2} = \cdot 8660\ldots, \quad \tan 30^\circ = \frac{1}{\sqrt{3}} = \cdot 577\ldots$$

86. In the following examples the student should notice

(i) the *angle* referred to,

(ii) that there is a *right angle* in the same triangle as the angle referred to,

(iii) the *perpendicular*, which is opposite the angle referred to, and is perpendicular to one of the lines containing the angle,

(iv) the *hypotenuse*, which is opposite the right angle,

(v) the *base*, the third side of the triangle.

Example. In the second figure on the next page, in which BDA is a right angle, find sin DBA and cos DBA.

In this case

(i) DBA is the *angle.*

(ii) BDA is a *right angle* in the same triangle as the angle DBA.

(iii) DA is the *perpendicular*, for it is opposite DBA and is perpendicular to BD.

(iv) BA is the *hypotenuse.*

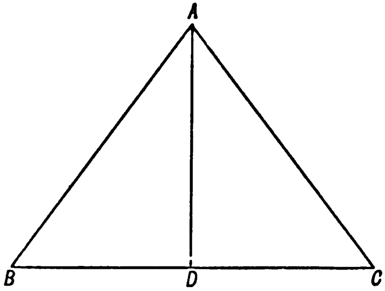
(v) BD is the *base.*

$$\text{Therefore}\quad \sin DBA,\text{ which is } \frac{\text{perpendicular}}{\text{hypotenuse}}, = \frac{DA}{BA},$$

$$\cos DBA,\text{ which is } \frac{\text{base}}{\text{hypotenuse}}, = \frac{BD}{BA}.$$

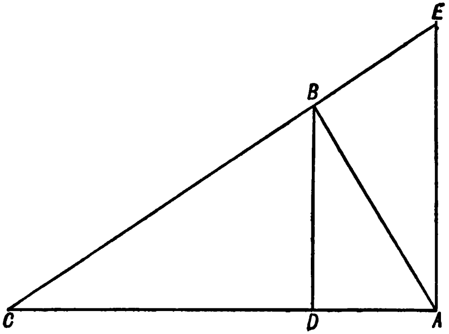
EXAMPLES. XII.

(1) Let ABC be any triangle and let AD be drawn perpendicular to BC. Write down the *perpendicular*, and the *base* when the following angles are referred to: (i) the angle ABD, (ii) the angle BAD, (iii) the angle ACD, (iv) the angle DAC.



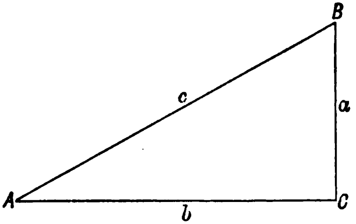
(2) Write down the following ratios in the above figure; (i) $\sin BAD$, (ii) $\cos ACD$, (iii) $\tan DAC$, (iv) $\sin ABD$, (v) $\tan BAD$, (vi) $\sin DAC$, (vii) $\cos DCA$, (viii) $\tan DCA$, (ix) $\cos ABD$, (x) $\sin ACD$.

(3) Let ACB be any angle and let ABC and BDC be right angles; (see next figure). Write down *two* values for each of the following ratios; (i) $\sin ACB$, (ii) $\cos ACB$, (iii) $\tan ACB$, (iv) $\sin BAC$, (v) $\cos BAC$, (vi) $\tan BAC$.



(4) In the accompanying figure BDC, CBA and EAC are right angles. Write down (i) $\sin DBA$, (ii) $\sin BEA$, (iii) $\sin CBD$, (iv) $\cos BAE$, (v) $\cos BAD$, (vi) $\cos CBD$, (vii) $\tan BCD$, (viii) $\tan DBA$, (ix) $\tan BEA$, (x) $\tan CBD$, (xi) $\sin DAB$, (xii) $\sin BAE$.

(5) Let ABC be a right-angled triangle such that $AB=5$ ft., $BC=3$ ft., then AC will be 4 ft.



Find the sine, cosine and tangent of the angles at A and B respectively.

In the above triangle if A stand for the angle at A and B for the angle at B, show that $\sin^2 A+\cos^2 A=1$, and that $\sin^2 B+\cos^2 B=1$.

(6) If ABC be any right-angled triangle with a right angle at C, and let A, B, and C stand for the angles at A, B and C respectively, and let a, b and c be the measures of the sides opposite the angles A, B and C respectively.

Show that $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$, $\tan A=\frac{a}{b}$

Show also that $\sin^2 A+\cos^2 A=1$.

Show also that (i) $a=c\,.\sin A$, (ii) $b=c\,.\sin B$, (iii) $a=c\,.\cos B$, (iv) $b=c\,.\cos A$, (v) $\sin A=\cos B$, (vi) $\cos A=\sin B$, (vii) $\tan A=\cot B$.

(7) The sides of a right-angled triangle are in the ratio 5 : 12 : 13. Find the sine, cosine and tangent of each acute angle of the triangle.

(8) The sides of a right-angled triangle are in the ratio 1 : 2 : $\sqrt{3}$. Find the sine, cosine and tangent of each acute angle of the triangle.

(9) Prove that if A be either of the angles of the above two triangles $\sin^2 A+\cos^2 A=1$.

CHAPTER VI.

ON THE TRIGONOMETRICAL RATIOS OF CERTAIN ANGLES.

87. The Trigonometrical Ratios of an angle are *numerical quantities simply*, as their name ratio implies. They are in nearly all cases incommensurable numbers.

Their practical value has been found for all angles between 0 and 90°, which differ by 1′; and a list of these values will be found in any volume of Mathematical Tables.

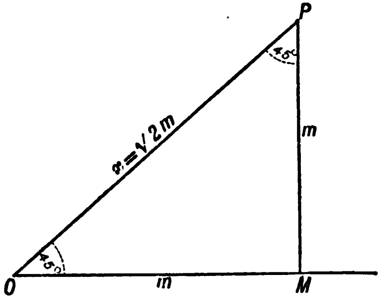
The student is recommended to get a copy of Chambers' Mathematical Tables for instruction and reference.

88. The finding the values of these Ratios has involved a large amount of labour; but, as the results have been published in Tables, the finding the Trigonometrical Ratios does not form any part of a student's work, except to exemplify the method employed.

89. The general method of finding Trigonometrical Ratios belongs to a more advanced part of the subject than the present, but there are certain angles whose Ratios can be found in a simple manner.

90. *To find the sine, cosine and tangent of an angle of* 45°.

If one angle of a right-angled triangle be 45°, that is, the half of a right angle, the third angle must also be 45°. Hence 45° is one angle of an isosceles right-angled triangle.



Let POM be an isosceles triangle such that PMO is a right angle, and $OM=MP$. Then $POM=OPM=45°$.

Let the measures of OM and of MP each be m. Let the measure of OP be x.

Then $$x^2=m^2+m^2=2m^2;$$
$$\therefore\ x=\sqrt{2}\,.\,m.$$

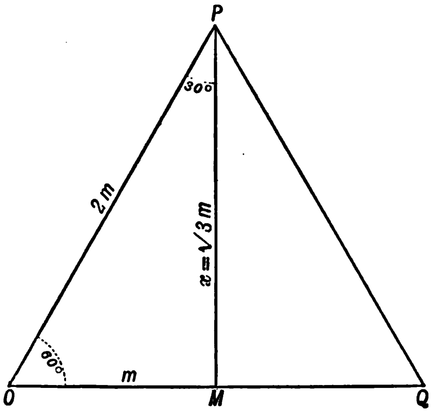
Hence, $$\sin 45° = \sin POM = \frac{MP}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$
$$\cos 45° = \cos POM = \frac{OM}{OP} = \frac{m}{\sqrt{2}\,.\,m} = \frac{1}{\sqrt{2}},$$
$$\tan 45° = \tan POM = \frac{MP}{OM} = \frac{m}{m} = \frac{1}{1} = 1.$$

91. *To find the sine, cosine and tangent of* 60°.

In an equilateral triangle, each of the equal angles is 60°, because they are each one third of 180°. And if we draw a perpendicular from one of the angular points of the triangle to the opposite side, we get a right-angled triangle in which one angle is 60°.

Let OPQ be an equilateral triangle. Draw PM perpendicular to OQ. Then OQ is bisected in M.

Let the measure of OM be m; then that of OQ is $2m$ and therefore that of OP is $2m$.



Let the measure of MP be x.

Then $$x^2 = (2m)^2 - m^2 = 4m^2 - m^2 = 3m^2,$$
$$\therefore\ x = \sqrt{3}\,.\,m.$$

Hence, $$\sin 60^\circ = \sin POM = \frac{MP}{OP} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$
$$\cos 60^\circ = \cos POM = \frac{OM}{OP} = \frac{m}{2m} = \frac{1}{2},$$
$$\tan 60^\circ = \tan POM = \frac{MP}{OM} = \frac{\sqrt{3}\,.\,m}{m} = \frac{\sqrt{3}}{1} = \sqrt{3}.$$

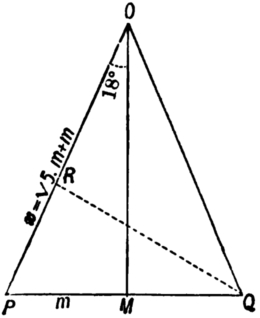
92. *To find the sine, cosine and tangent of* 30°.

With the same figure and construction as above, we have the angle $OPM = 30^\circ$, since it is a half of OPQ, i.e. of 60°.

Hence, $$\sin 30^\circ = \sin OPM = \frac{MO}{PO} = \frac{m}{2m} = \frac{1}{2},$$
$$\cos 30^\circ = \cos OPM = \frac{PM}{PO} = \frac{\sqrt{3}\,.\,m}{2m} = \frac{\sqrt{3}}{2},$$
$$\tan 30^\circ = \tan OPM = \frac{MO}{PM} = \frac{m}{\sqrt{3}\,.\,m} = \frac{1}{\sqrt{3}}.$$

**93. *To find the sine of* 18°.

In the 10th Prop. of Euclid IV. a triangle is described such that each of the angles at the base is double of the third angle. [See also Example (6) p. 140.]



Let POQ be such a triangle, and let the vertical angle POQ contain n degrees; then

$$n + 2n + 2n = 180,$$

$$n = 36.$$

Draw OM perpendicular to PQ, bisecting the angle POQ. Then since $QOP = 36°$, $MOP = 18°$. Also $PM = MQ$.

Let the measure of MP be m, and the measure of OP be x. From OP cut off $OR = PQ$. Then by Euclid IV. 10 $PO \,.\, PR = PQ^2$.

$$\therefore\ x(x - 2m) = (2m)^2,$$

$$\therefore\ x^2 - 2mx + m^2 = 4m^2 + m^2 = 5m^2,$$

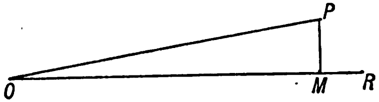
$$\therefore\ x - m = \sqrt{5}\,.\,m,$$

$$\therefore\ x = \sqrt{5}\,.\,m + m,$$

$$\therefore\ \sin 18° = \sin MOP = \frac{MP}{OP} = \frac{m}{\sqrt{5}\,.\,m + m} = \frac{1}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{4}.$$

94. *To find the sine, cosine and tangent of* 0°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of a very small angle approach, as the angle is continually diminished.*



Let ROP be a small angle. Draw PM perpendicular to OR, and let OP be always of the same length, so that P lies on a circle whose centre is O.

Then if the angle ROP be diminished, we can see that MP is diminished also, and that consequently $\frac{MP}{OP}$, which is sin ROP, is diminished. And, by diminishing the angle ROP sufficiently, *we can make* MP *as small as we please*, and therefore *we can make* sin ROP *smaller than any assignable number however small that number may be.*

Thus we see that the value to which sin ROP approaches as the angle is diminished, is 0.

This is expressed by saying, $\sin 0° = 0$................. i.

Again, as the angle ROP diminishes, OM approaches OP in length; and cos ROP, which is $\frac{OM}{OP}$, approaches in value to $\frac{OP}{OP}$, i.e. to 1.

This is expressed by saying, $\cos 0° = 1$ .... ii.

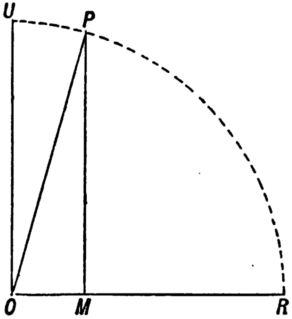
Also, tan ROP is $\frac{MP}{OM}$; and we have seen that MP approaches 0, while OM does not; $\therefore$ tan ROP approaches 0.

This is expressed by saying, $\tan 0° = 0$ .... iii.

95. *To find the sine, cosine and tangent of* 90°.

By this is meant,—*To find the values, if any, to which the Trigonometrical Ratios of an angle approach, as the angle approaches a right angle.*

Let ROU be a right angle $= 90°$.



Draw ROP nearly a right angle; draw PM perpendicular to OR, and let OP be always of the same length, so that P lies on a circle whose centre is O.

Then, as the angle ROP approaches to ROU, we can see that MP approaches OP, while OM continually diminishes.

Hence when ROP approaches 90°, sin ROP, which is $\frac{MP}{OP}$, approaches in value to $\frac{OP}{OP}$, that is to $\frac{1}{1}$, i.e. to 1.

Hence we say, that $\sin 90° = 1$ .... i.

Again, when ROP approaches 90°, cos ROP, which is $\frac{OM}{OP}$, approaches in value to $\frac{0}{OP}$, that is to 0.

Hence we say, that $\cos 90° = 0$ .... ii.

Again, when *ROP* approaches 90°, tan *ROP* which is $\frac{MP}{OM}$ approaches in value to $\frac{OP}{\text{a quantity which approaches } 0}$.

But in any fraction whose numerator does not diminish, the smaller the denominator the greater is the value of that fraction; and if the denominator continually diminishes the value of the fraction continually increases.

Hence, *tan ROP can be made larger than any assignable number by making the angle ROP approach* 90° *near enough.*

This is what we mean when we say, that

tan 90° is **infinity**, or, tan $90^\circ = \infty$ .... iii.

96. The following table exhibits the results of this Chapter.

angle	0°	18°	30°	45°	60°	90°
sine	0	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1		$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0		$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

The student may notice that the sine increases with the angle, while the cosine diminishes as the angle increases.

Also that the squares of the sines of 0°, 30°, 45°, 60° and 90° are respectively 0, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$ and $\frac{4}{4}$, and that the squares of the cosines of the same angles are $\frac{4}{4}$, $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$, and 0.

EXAMPLES. XIII.

If $A=90^0, B=60^0, C=30^0, D=45^0, E=18^0$, prove the following:

(1) $2.\sin D.\cos D=\sin A$. (2) $2.\sin C.\cos C=\sin B$.

(3) $\cos^2 B-\sin^2 B=1-2\sin^2 B$.

(4) $\sin B.\cos C+\sin C.\cos B=\sin A$.

(5) $\cos^2 D-\sin^2 D=\cos A$. (6) $4.\sin^2 E+2.\sin E=1$.

(7) $\sin^2 B+\cos^2 B=1$. (8) $\cos^2 C+\sin^2 C=1$.

(9) $\cos^2 D+\sin^2 D=1$.

(10) $\sin B.\cos C-\sin C.\cos B=\sin C$.

(11) $2(\cos B.\cos D+\sin B.\sin D)^2=1+\cos C$.

(12) $2(\sin D.\cos C-\sin C.\cos D)^2=1-\cos C$.

(13) $\sin 30^0=\cdot 5$. (14) $\sin 45^0=\cdot 7071\ldots$

(15) $\sin 60^0=\cdot 8660\ldots$ (16) $\tan 60^0=1\cdot 7320508\ldots$

(17) $\tan 30^0=\cdot 5773\ldots$ (18) $\sin 18^0=\cdot 3090\ldots$

97. The actual measurement of the *line* joining two points which are any considerable distance apart, is a very tedious and difficult operation, especially when great accuracy is required; while the accurate measurement of an *angle* can, with proper instruments, be made with comparative ease and quickness.

98. A **Sextant** is an instrument for measuring the angle between the two lines drawn from the observer's eye to each of two distant objects respectively.

A **Theodolite** is an instrument for measuring angles in a horizontal plane; also for measuring '*angles of elevation*' and '*angles of depression.*'

99. The angle made with the horizontal plane, by the line joining the observer's eye with a distant object, is called

(i) its **angle of elevation**, when the object is *above* the observer;

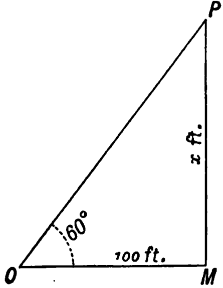
(ii) its **angle of depression**, when the object is *below* the observer.†

100. **Trigonometry** enables us by measuring certain *angles*, to deduce, from one known distance, the *lengths* of other distances: or, by the measurement of a *convenient* line, to deduce by the measurement of *angles* the lengths of lines whose actual measurement is difficult or impossible.

[In the Trigonometrical Survey of England, made by the Ordnance Department, the only distance actually measured was one of about seven miles on Salisbury Plain.]

101. For this purpose we require the numerical values of the Trigonometrical Ratios of the angles observed. Accordingly mathematical tables have been compiled, containing *lists* of the values of these Ratios. These Tables constitute a kind of numerical **Dictionary**, in which we can find the numerical value of the Trigonometrical Ratios of any required angle.

Example 1. At a point 100 feet from the foot of a tower the angle of elevation of the top of the tower is observed to be 60°. Find the height above the point of observation of the top of the tower.



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† In measuring the angle of *depression* the telescope is turned from a horizontal position *downwards*. See Ex. (3) p. 63.

Let O be the point of observation; let P be the top of the tower; let a horizontal line through O meet the foot of the tower at the point M. Then $OM = 100$ feet, and the angle $MOP = 60^0$. Let MP contain x feet.

Then
$$\frac{MP}{OM} = \tan MOP = \tan 60^0 = \sqrt{3}$$

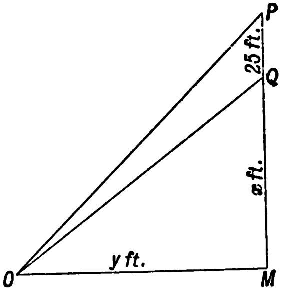
$$\therefore \frac{x}{100} = \sqrt{3}.$$

$$\therefore x = 100 . \sqrt{3} = 100 \times 1{\cdot}7320 \text{ etc.}$$

$$= 173{\cdot}2.$$

Therefore the required height is 173·2 feet.

Example 2. A flagstaff, 25 feet high, stands on the top of a cliff, and from a point on the seashore the angles of elevation of the highest and lowest points of the flagstaff are observed to be $47^0\ 12'$ and $45^0\ 13'$ respectively. Find the height of the cliff.



Let O be the point of observation, PQ the flagstaff.

Let a horizontal line through O meet the vertical line PQ produced in M.

Then $QP = 25$ feet, $MOP = 47^0\ 12'$, $MOQ = 45^0\ 13'$.

Let $MQ = x$ feet; let $OM = y$ feet.

Then $\frac{MP}{OM} = \tan 47^0\ 12'$, $\therefore\ \frac{x+25}{y} = \tan 47^0\ 12'$,

and $\frac{MQ}{OM} = \tan 45^0\ 13'$, $\therefore\ \frac{x}{y} = \tan 45^0\ 13'$.

Hence, by division $\therefore\ \frac{x+25}{x} = \frac{\tan 47^0\ 12'}{\tan 45^0\ 13'}$.

In the Tables we find that

$$\tan 47^0\ 12' = 1{\cdot}0799018, \text{ and } \tan 45^0\ 13' = 1{\cdot}0075918,$$

$$\therefore\ 1 + \frac{25}{x} = \frac{1{\cdot}0799018}{1{\cdot}0075918} = 1 + \frac{{\cdot}0723100}{1{\cdot}0075918},$$

$$\therefore\ \frac{x}{25} = \frac{1{\cdot}0075918}{{\cdot}0723100} = \frac{100759}{7231}.$$

$$\therefore\ x = \frac{2518975}{7231} = 348 \text{ nearly.}$$

Therefore the cliff is 348 feet high.

EXAMPLES. XIV.

The answers are given correct to three significant figures.

(1) At a point 179 feet in a horizontal line from the foot of a column, the angle of elevation of the top of the column is observed to be 45°. What is the height of the column ?

(2) At a point 200 feet from, and on a level with the base of a tower, the angle of elevation of the top of the tower is observed to be 60°: what is the height of the tower?

(3) From the top of a vertical cliff, the angle of depression of a point on the shore 150 feet from the base of the cliff, is observed to be 30°: find the height of the cliff.

(4) From the top of a tower 117 feet high the angle of depression of the top of a house 37 feet high is observed to be 30°: how far is the top of the house from the tower?

(5) A man 6 ft. high stands at a distance of 4 ft. 9 in. from a lamp-post, and it is observed that his shadow is 19 ft. long. Find the height of the lamp.

(6) The shadow of a tower in the sunlight is observed to be 100 ft. long, and at the same time the shadow of a lamp-post 9 ft. high is observed to be $3\sqrt{3}$ ft. long. Find the angle of elevation of the sun, and the height of the tower.

(7) From a point P on the bank of a river, just opposite a post Q on the other bank, a man walks at right angles to PQ to a point R so that PR is 100 yards; he then observes the angle PRQ to be $32^0\ 17'$: find the breadth of the river.

$$\tan 32^0\ 17' = \cdot 6317667.$$

(8) A flagstaff 25 feet high stands on the top of a house; from a point on the plain on which the house stands the angles of elevation of the top and bottom of the flagstaff are observed to be 60^0 and 45^0 respectively: find the height of the house above the point of observation.

(9) From the top of a cliff 100 feet high, the angles of depression of two ships at sea are observed to be 45^0 and 30^0 respectively; if the line joining the ships points directly to the foot of the cliff, find the distance between the ships.

(10) A tower 100 feet high stands on the top of a cliff; from a point on the sand at the foot of the cliff the angles of elevation of the top and bottom of the tower are observed to be 75^0 and 60^0 respectively; find the height of the cliff. (Tan $75^0 = 2 + \sqrt{3}$).

(11) A man walking along a straight road observes at one milestone a house in a direction making an angle 30^0 with the road, and that at the next milestone the angle is 60^0: how far is the house from the road?

(12) A man stands at a point A on the bank AB of a straight river and observes that the line joining A to a post C on the opposite bank makes with AB an angle of 30^0. He then goes 400 yards along the bank to B and finds that BC makes with BA an angle of 60^0; find the breadth of the river.

**(13) A building on a square base *ABCD* has two of its sides, *AB* and *CD*, parallel to the bank of a river. An observer, standing at *E* on the other side of the river so that *DAE* is a straight line, finds that *AB* subtends at his eye an angle of 45^0. Having walked *a* yards parallel to the bank, he finds that *DE* subtends an angle whose tangent is $\sqrt{2}$. Show that $DB = a$ yards.

(14) From the top of a hill the angles of depression of the top and bottom of a flagstaff 25 feet high at the foot of the hill are observed to be $45^0\,13'$ and $47^0\,12'$ respectively; find the height of the hill.

$$\tan 45^0\,13' = 1{\cdot}0075918,$$
$$\tan 47^0\,12' = 1{\cdot}0799018.$$

(15) From each of two stations, East and West of each other, the altitude of a balloon is observed to be 45^0, and its bearings to be respectively N.W. and N.E.: if the stations be 1 mile apart, determine the height of the balloon.

(16) The angle of elevation of a balloon from a station due south of it is 60^0; and from another station due west of the former and distant a mile from it it is 45^0. Find the height of the balloon.

(17) An isosceles triangle of wood is placed on the ground in a vertical position facing the sun. If $2a$ be the base of the triangle, *b* its height, and 30^0 the altitude of the sun, find the tangent of half the angle at the apex of the shadow.

(18) The length of the shadow of a vertical stick is to the length of the stick as $\sqrt{3} : 1$. If the stick be turned about its lower extremity in a vertical plane, so that the shadow is always in the same direction, find what will be the angle of its inclination to the horizon when the length of the shadow is the same as before.

(19) What distance in space is travelled in an hour in consequence of the earth's rotation, by a person situated in latitude 60^0? (Earth's radius = 4000 miles.)

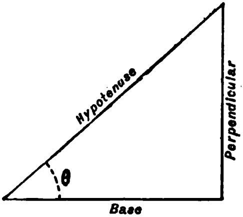
CHAPTER VII.

ON THE RELATIONS BETWEEN THE TRIGONOMETRICAL RATIOS OF THE SAME ANGLE.

102. THE following relations are evident from the definitions:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}, \qquad \sec\theta = \frac{1}{\cos\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

103. To prove $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.



We have $\sin\theta = \dfrac{\text{perpendicular}}{\text{hypotenuse}}$,

and $\cos\theta = \dfrac{\text{base}}{\text{hypotenuse}}$;

$$\therefore \frac{\sin\theta}{\cos\theta} = \frac{\text{perpendicular}}{\text{base}} = \tan\theta.$$

104. We may prove similarly $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Or thus, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$.

105. Euclid I. 47 gives us that in any right-angled triangle the square on the hypotenuse = the sum of the squares on the perpendicular and on the base,

$$\text{or, } (\text{hypotenuse})^2 = (\text{perpendicular})^2 + (\text{base})^2.$$

(i) Divide each side of this identity by (hypotenuse)^2, and we get

$$\left(\frac{\text{hypotenuse}}{\text{hypotenuse}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{hypotenuse}}\right)^2 + \left(\frac{\text{base}}{\text{hypotenuse}}\right)^2,$$

that is, $$1 = \sin^2\theta + \cos^2\theta.$$

(ii) Divide each side of the same identity by (base)^2, and we get

$$\left(\frac{\text{hypotenuse}}{\text{base}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{base}}\right)^2 + \left(\frac{\text{base}}{\text{base}}\right)^2,$$

that is, $$\sec^2\theta = \tan^2 + 1.$$

(iii) Divide each side of the same identity by (perpendicular)^2, and we get

$$\left(\frac{\text{hypotenuse}}{\text{perpendicular}}\right)^2 = \left(\frac{\text{perpendicular}}{\text{perpendicular}}\right)^2 + \left(\frac{\text{base}}{\text{perpendicular}}\right)^2,$$

that is, $$\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$$

106. Thus the three results

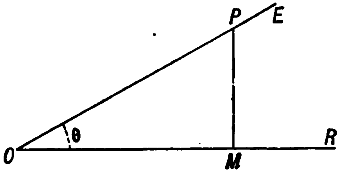
$$\left.\begin{array}{ll} \text{(i)} & \cos^2\theta + \sin^2\theta = 1 \\ \text{(ii)} & 1 + \tan^2\theta = \sec^2\theta \\ \text{(iii)} & 1 + \cot^2\theta = \operatorname{cosec}^2\theta \end{array}\right\},$$

are each a statement in Trigonometrical language of Euc. I. 47.

107. We give the above proof in a different form.

To prove that $\cos^2\theta + \sin^2\theta = 1$.

Let ROE be any angle θ.



In OE take any point P, and draw PM perpendicular to OR. Then with respect to θ, MP is the perpendicular, OP is the hypotenuse, and OM is the base;

$$\therefore \ \sin^2\theta = \frac{MP^2}{OP^2}, \quad \cos^2\theta = \frac{OM^2}{OP^2}.$$

We have to prove that $\sin^2\theta + \cos^2\theta = 1$,

$$\text{that is, that } \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = 1,$$

$$\textit{i.e.} \text{ that } \frac{MP^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2},$$

$$\textit{i.e.} \text{ that } MP^2 + OM^2 = OP^2.$$

But this is true by Euclid I. 47.

Therefore $\cos^2\theta + \sin^2\theta = 1$.

Similarly we may prove that

$$1 + \tan^2\theta = \sec^2\theta,$$

and that $$1 + \cot^2\theta = \operatorname{cosec}^2\theta.$$

108. The following is a LIST OF FORMULÆ with which the student must make himself familiar:

$$\operatorname{cosec}\theta=\frac{1}{\sin\theta},$$

$$\sec\theta=\frac{1}{\cos\theta},$$

$$\cot\theta=\frac{1}{\tan\theta},$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta},$$

$$\cot\theta=\frac{\cos\theta}{\sin\theta},$$

$$\sin^2\theta+\cos^2\theta=1,$$

$$\tan^2\theta+1=\sec^2\theta,$$

$$\cot^2\theta+1=\operatorname{cosec}^2\theta.$$

109. In proving Trigonometrical identities it is often convenient to express the other Trigonometrical Ratios *in terms of the sine and cosine.*

Example. Prove that tan A + *cot* A = *sec* A . *cosec* A.

Since $\tan A=\dfrac{\sin A}{\cos A}$, $\cot A=\dfrac{\cos A}{\sin A}$,

$$\sec A=\frac{1}{\cos A}\text{ and }\operatorname{cosec}A=\frac{1}{\sin A},$$

we have to prove that

$$\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{1}{\cos A}\cdot\frac{1}{\sin A},$$

or that

$$\frac{\sin^2 A+\cos^2 A}{\cos A\,.\sin A}=\frac{1}{\cos A\,.\sin A},$$

and this is true, because $\sin^2 A+\cos^2 A=1$.

110. Sometimes it is more convenient to express all the other Trigonometrical Ratios in terms of the *sine* only, or in terms of the *cosine* only.

Example. Prove that $\sin^4\theta + 2\sin^2\theta \,.\, \cos^2\theta = 1 - \cos^4\theta$.

Now since $\sin^2\theta + \cos^2\theta = 1$,

therefore $\sin^2\theta = 1 - \cos^2\theta$,

and therefore $\sin^4\theta = (1 - \cos^2\theta)^2$.

Hence, putting $1 - \cos^2\theta$, and $(1 - \cos^2\theta)^2$ for $\sin^2\theta$ and $\sin^4\theta$ respectively, we have to prove that

$$(1 - \cos^2\theta)^2 + 2\,.\,(1 - \cos^2\theta)\,.\,\cos^2\theta = 1 - \cos^4\theta,$$

or that

$$1 - 2\cos^2\theta + \cos^4\theta + 2\cos^2\theta - 2\cos^4\theta = 1 - \cos^4\theta,$$

or that $1 - \cos^4\theta = 1 - \cos^4\theta$,

which is true.

This example may be proved *directly*, by reversing the steps of the above proof; thus

We have $1 - \cos^4\theta = 1 - \cos^4\theta$,

$$\therefore\ (1 - 2\cos^2\theta + \cos^4\theta) + 2\cos^2\theta - 2\cos^4\theta = 1 - \cos^4\theta,$$

$$\therefore\ (1 - \cos^2\theta)^2 + 2\cos^2\theta\,(1 - \cos^2\theta) = 1 - \cos^4\theta,$$

$$\therefore\ (\sin^2\theta)^2 + 2\cos^2\theta\,.\,\sin^2\theta = 1 - \cos^4\theta. \quad \text{Q.E.D.}$$

NOTE. $(1 - \cos\theta)$ is called the **versed sine** of θ; it is abbreviated thus versin θ.

EXAMPLES. XV.

Prove the following statements.

(1) $\cos A \,.\, \tan A = \sin A$, (2) $\cot A \,.\, \tan A = 1$.

(3) $\cos A = \sin A \,.\, \cot A$. (4) $\sec A \,.\, \cot A = \operatorname{cosec} A$.

(5) $\operatorname{cosec} A \,.\, \tan A = \sec A$.

(6) $(\tan A + \cot A) \sin A \,.\, \cos A = 1$.

(7) $(\tan A - \cot A) \sin A \,.\, \cos A = \sin^2 A - \cos^2 A$.

(8) $\cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$.

(9) $(\sin A + \cos A)^2 = 1 + 2\sin A \,.\, \cos A$.

(10) $(\sin A - \cos A)^2 = 1 - 2\sin A \,.\, \cos A$.

(11) $\cos^4 B - \sin^4 B = 2\cos^2 B - 1$.

(12) $(\sin^2 B + \cos^2 B)^2 = 1$.

(13) $(\sin^2 B - \cos^2 B)^2 = 1 - 4\cos^2 B + 4\cos^4 B$.

(14) $1 - \tan^4 B = 2\sec^2 B - \sec^4 B$.

(15) $(\sec B - \tan B)(\sec B + \tan B) = 1$.

(16) $(\operatorname{cosec} \theta - \cot \theta)(\operatorname{cosec} \theta + \cot \theta) = 1$.

(17) $\sin^3 \theta + \cos^3 \theta = (\sin \theta + \cos \theta)(1 - \sin \theta \cos \theta)$.

(18) $\cos^3 \theta - \sin^3 \theta = (\cos \theta - \sin \theta)(1 + \sin \theta \cos \theta)$.

(19) $\sin^6 \theta + \cos^6 \theta = 1 - 3\sin^2 \theta \,.\, \cos^2 \theta$.

(20) $(\sin^6 \theta - \cos^6 \theta) = (2\sin^2 \theta - 1)(1 - \sin^2 \theta + \sin^4 \theta)$.

(21) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \,.\, \tan B$.

(22) $\dfrac{\cot \alpha + \tan \beta}{\tan \alpha + \cot \beta} = \cot \alpha \,.\, \tan \beta$.

(23) $\dfrac{1 - \sin A}{1 + \sin A} = (\sec A - \tan A)^2$.

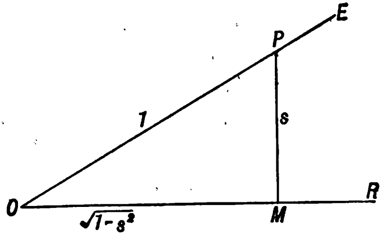
(24) $\dfrac{1 + \cos A}{1 - \cos A} = (\operatorname{cosec} A + \cot A)^2$.

(25) $2 \operatorname{versin} \theta - \operatorname{versin}^2 \theta = \sin^2 \theta$.

(26) $\operatorname{versin} \theta \,(1 + \cos \theta) = \sin^2 \theta$.

111. All the Trigonometrical Ratios of an angle can be expressed in terms of any one of them.

Example 1. *To express all the trigonometrical ratios of an angle in terms of the sine.*



Let ROE be any angle A.

We can take P anywhere in the line OE; so that we can make *one* of the lines, OP, OM, or MP any length we please.

Let us take OP so that its measure is 1, and let s be the measure of MP; so that $\sin A$, which is $\frac{MP}{OP}$, $=\frac{s}{1}$; or, $s=\sin A$.

Let x be the measure of OM.

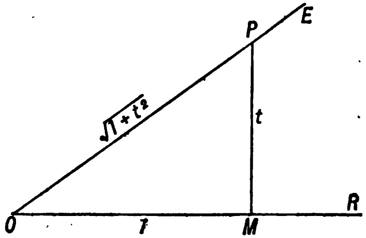
Then since $$OM^2=OP^2-MP^2,$$
$$\therefore\ x^2=1-s^2,$$
$$\therefore\ x=\sqrt{1-s^2},$$

Hence $$\cos A=\frac{OM}{OP}=\frac{\sqrt{1-s^2}}{1}=\sqrt{1-\sin^2 A},$$
$$\tan A=\frac{MP}{OM}=\frac{s}{\sqrt{1-s^2}}=\frac{\sin A}{\sqrt{1-\sin^2 \theta}},$$

and so on.

Note. The solution of the equation $x^2=1-s^2$, gives
$$x=\pm\sqrt{1-s^2},$$
and therefore the ambiguity (±) must stand before each of the root symbols in the above. This ambiguity, as will be explained later on, is of great use when the magnitude of the angle is not limited. When we limit A to be less than a right angle we have no use for the negative sign.

Example 2. To express all the other trigonometrical ratios of an angle in terms of the tangent.



In this case $\tan\theta = \frac{MP}{OM}$.

Take P so that the measure of OM is 1, and let t be the measure of MP; so that $\tan\theta$, which is $\frac{MP}{OM}$, $=\frac{t}{1}$; or, $t = \tan\theta$.

Then we can show that the measure of OP is $\sqrt{1+t^2}$.

Hence, $$\sin\theta = \frac{MP}{OP} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

$$\cos\theta = \frac{OM}{OP} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}}.$$

and so on.

112. The same results may be obtained by the use of the formulæ on page 69.

Example $$\cos^2\theta + \sin^2\theta = 1,$$

$$\therefore \cos^2\theta = 1 - \sin^2\theta,$$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta}.$$

Again $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}},$$

and so on.

EXAMPLES. XVI.

(1) Express all the other Trigonometrical Ratios of A in terms of $\cos A$.

(2) Express all the other Trigonometrical Ratios of A in terms of $\cot A$.

(3) Express all the other Trigonometrical Ratios of A in terms of $\sec A$.

(4) Express all the other Trigonometrical Ratios of A in terms of $\operatorname{cosec} A$.

(5) Use the formulæ of page 69 to express all the other Trigonometrical Ratios of A in terms of $\sin A$.

(6) Use the formulæ of page 69 to express all the other Trigonometrical Ratios of A in terms of the $\tan A$.

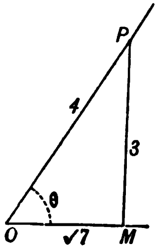
113. *Given one of the Trigonometrical Ratios of an angle less than a right angle, we can find all the others.*

Since all the Trigonometrical Ratios of an angle can be expressed in terms of any one of them, it is clear that if the numerical value of any one of them be given, the numerical value of all the rest can be found.

Example. Given $\sin \theta = \frac{3}{4}$, find the other Trigonometrical Ratios of θ.

Let ROE be the angle θ.

Take P on OE so that the measure of OP is 4. Draw PM perpendicular to OR.



Then since $\sin\theta=\frac{3}{4}$ $\left(\text{so that } \frac{MP}{OP}=\frac{3}{4}\right)$, and since the measure of OP is 4, therefore the measure of MP must be 3.

Let x be the measure of OM;

then
$$OM^2=OP^2-MP^2,$$
$$\therefore\ x^2=4^2-3^2=16-9=7.$$
$$x=\sqrt{7}.$$

Therefore the measure of OM is $\sqrt{7}$.

Hence,
$$\cos\theta=\frac{OM}{OP}=\frac{\sqrt{7}}{4},$$
$$\tan\theta=\frac{MP}{OM}=\frac{3}{\sqrt{7}}=\frac{3\sqrt{7}}{7}.$$
$$\cot\theta=\frac{\sqrt{7}}{3}.$$

EXAMPLES. XVII.

(1) Given that $\sin A=\frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.

(2) Given that $\cos B=\frac{1}{3}$, find $\sin B$ and $\cot B$.

(3) Given that $\tan A=\frac{4}{3}$, find $\sin A$ and $\sec A$.

(4) Given that $\sec\theta=4$, find $\cot\theta$ and $\sin\theta$.

(5) Given that $\tan\theta=\sqrt{3}$, find $\sin\theta$ and $\cos\theta$.

(6) Given that $\cot\theta=\frac{2}{\sqrt{5}}$, find $\sin\theta$ and $\sec\theta$.

(7) Given that $\sin\theta=\frac{b}{c}$, find $\tan\theta$.

(8) Given that $\tan\theta=\frac{a}{b}$, find $\sin\theta$ and $\cos\theta$.

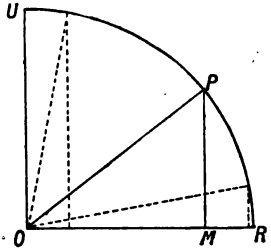
(9) Given that $\cos\theta=\frac{1}{a}$, find $\sin\theta$ and $\cot\theta$.

(10) If $\sin\theta=a$, and $\tan\theta=b$, prove that $(1-a^2)(1+b^2)=1$.

(11) If $\cos\theta=h$, and $\tan\theta=k$, find the equation connecting h and k.

114. The following examples are important.

Example 1. To trace the changes in the magnitude of sin A as A increases from 0^0 to 90^0.



Take a line OR, of any length; and describe the quadrant RPU of the circle whose centre is O and radius OR.

Draw the right angle ROU, cutting the circle in U.

Let OP make any angle ROP $(=A)$ with OR; draw PM perpendicular to OR.

Then $$\sin A = \frac{MP}{OP}.$$

When the angle A is 0^0, MP is zero, and when A is 90^0, MP is equal to OP; and as A continuously increases from 0^0 to 90^0, MP increases continuously from zero to OP; also OP is always equal to OR.

Therefore, when $A=0^0$, the fraction $\frac{MP}{OP}$ is equal to $\frac{0}{OP}$, that is 0; when $A=90^0$ the fraction $\frac{MP}{OP}$ is equal to $\frac{OP}{OP}$, that is 1; and as A continuously increases from 0^0 to 90^0, the *numerator* of the fraction $\frac{MP}{OP}$ *continuously increases* from zero to OP, while the denominator is unchanged, and therefore the *fraction* $\frac{MP}{OP}$, which is sin A, *increases continuously* from 0 to 1.

Example 2. To trace the changes in the magnitude of tan A as A increases from 0^0 to 90^0.

With the same construction and figure as in the last article, we have

$$\tan A = \frac{MP}{OM}.$$

When the angle A is 0^0, MP is zero; when A is 90^0, MP is equal to OP; and as the angle continuously increases from 0^0 to 90^0, MP increases continuously from zero to OP.

When the angle A is 0^0, OM is equal to OP; when A is 90^0, OM is zero; and as A continuously increases from 0^0 to 90^0, OM continuously decreases from OP to zero.

Hence, when A is 0^0, the fraction $\frac{MP}{OM}$ is equal to $\frac{0}{OM}$, that is 0; when A is 90^0, the fraction $\frac{MP}{OM}$ is equal to $\frac{OP}{0}$, that is 'infinity' (see art. 95); and as A continuously increases from 0^0 to 90^0, the numerator continuously increases from zero to OP, while the denominator continuously diminishes from OP to zero; so that the fraction $\frac{MP}{OM}$, which is tan A, *continuously* increases from 0 until it is *greater than any assignable numerical quantity.*

EXAMPLES. XVIII.

(1) Show that as A continuously increases from 0^0 to 90^0, cos A continuously diminishes from 1 to 0.

(2) Trace the changes in the magnitude of sec θ as θ increases from 0 to $\frac{\pi}{2}$.

(3) Trace the changes in the magnitude of sin A as A diminishes from 90^0 to 0^0.

(4) Trace the changes in the magnitude of cot θ as θ increases from 0 to $\frac{\pi}{2}$.

115. Since the hypotenuse is the *greatest* side in a right-angled triangle, it is clear

(i) that *sin* A, which is $\dfrac{\text{perpendicular}}{\text{hypotenuse}}$, is never greater than unity,

(ii) that *cos* A, which is $\dfrac{\text{base}}{\text{hypotenuse}}$, is never greater than unity,

(iii) that *cosec* A, which is $\dfrac{\text{hypotenuse}}{\text{perpendicular}}$, is never *numerically* less than unity (we shall see later on that it may be *negative*),

(iv) that *sec* A, which is $\dfrac{\text{hypotenuse}}{\text{base}}$, is never numerically less than unity (it may be negative).

Hence such a question as '*Find an angle whose* sine *is* $\frac{4}{3}$' has no solution, since there is no angle whose sine is greater than unity.

And such a question as '*Find an angle whose* secant *is* $\frac{3}{4}$' has no solution, because there is no angle whose secant is numerically less than unity.

116. If A be small, the perpendicular is smaller than the base; and tan A, which is $\dfrac{\text{perpendicular}}{\text{base}}$, can be made as small as we please (see figure in Art. 114). Also if the angle A be nearly a right angle, the perpendicular is greater than the base; and tan A, which is $\dfrac{\text{perpendicular}}{\text{base}}$, can be made as large as we please.

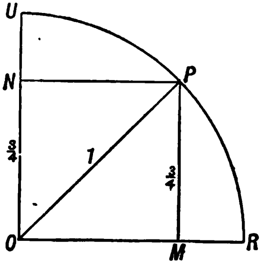
So that an angle whose **tangent** is a can *always* be found if a be positive; that is, there is always an angle A between 0° and 90°, such that $\tan A = a$, if a be a positive number.

[The student will find as he proceeds that the equation $\tan x = a$ can always be solved if a be any *arithmetical* quantity].

Example 1. *To draw an angle* A *whose sine is* $\frac{3}{4}$.

Draw any line OR, and take R such that the measure of OR is unity; describe the quadrant RPU of the circle whose centre is O and radius OR.

In OU which is perpendicular to OR, take ON so that the measure of ON is $\frac{3}{4}$.



Draw NP parallel to OR cutting the quadrant in P. (The student should observe that if the measure of ON were *greater than unity*, N would be outside the circle altogether, and the line parallel to OR would not cut the circle at all.)

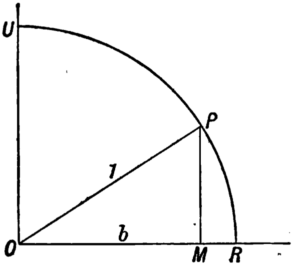
Join OP, and draw PM perpendicular to OR.

Then ROP is the angle required.

For $\sin ROP = \frac{MP}{OP} = \frac{ON}{OP} = \frac{3}{4} \div 1 = \frac{3}{4}$.

Therefore an angle POR has been drawn whose sine is $\frac{3}{4}$.

Example 2. *To draw an angle whose cosine is* b; where b is a *proper* fraction and positive.



Draw any line OR, and take OR so that its measure is unity.

Describe the quadrant RPU of the circle whose centre is O and radius OR.

Take M in OR so that the measure of OM is b. OM will be less than OR because b is a *proper* fraction.

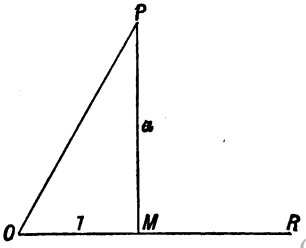
Draw MP perpendicular to OR cutting the quadrant in P. (The student will observe that if b were an *improper* fraction, M would be outside the circle altogether.)

Then the angle ROP is the angle required.

For $\cos ROP = \frac{OM}{OP} = \frac{b}{1} = b$.

Therefore an angle ROP has been described such that the $\cos ROP = b$.

Example 3. *To draw an angle A such that* $\tan A = a$, where a is any positive numerical quantity.



Draw any line OR, and take M in it so that the measure of OM is unity.

Draw MP perpendicular to OM, and take P so that the measure of MP is a. Join OP. Then the angle ROP is the angle required.

For $\tan ROP = \dfrac{MP}{OM} = \dfrac{a}{1} = a$.

Thus an angle ROP has been drawn so that $\tan ROP = a$.

117. Definition. One angle is called the **complement** of another, when the two angles added together make up a *right angle.*

Example 1. The complement of A is $(90^\circ - A)$.

Example 2. The complement of 190° is $(90^\circ - 190^\circ) = -100^\circ$. For $190^\circ + (90^\circ - 190^\circ) = 90^\circ$.

Example 3. The complement of $\dfrac{5\pi}{4}$ is $\left(\dfrac{\pi}{2} - \dfrac{5\pi}{4}\right) = -\dfrac{3\pi}{4}$.

118. *To prove that the sine of an angle* A *is equal to the cosine of its complement* $(90^\circ - \mathrm{A})$.

[This is true whatever be the magnitude of A, as will be proved later on.]

If A be less than 90°, let ROP be A (see last figure).

Draw PM perpendicular to OR.

Then since $PMO = 90^\circ$, therefore $POM + OPM = 90^\circ$ and therefore $OPM = (90^\circ - A)$.

Now, $\sin A = \dfrac{MP}{OP} = \cos OPM = \cos(90^\circ - A)$. Q.E.D.

EXAMPLES. XIX.

(1) Draw an angle whose sine is $\frac{1}{2}$.

(2) Draw an angle whose cosecant is 2.

(3) Draw an angle whose tangent is 2.

(4) Can an angle be drawn whose tangent is 431?

(5) Can an angle be drawn whose cosine is $\frac{4}{3}$?

(6) Can an angle be drawn whose secant is 5?

(7) Find the complements of

(i) 30^0. (ii) 190^0. (iii) 90^0. (iv) 350^0.

(v) -25^0. (vi) -320^0. (vii) $\frac{3\pi}{4}$. (viii) $-\frac{\pi}{6}$.

(8) Prove that $\sin 70^0 = \cos 20^0$.

(9) Prove that $\cos 47^0\,16' = \sin 42^0\,44'$.

(10) Prove that $\tan 79^0 = \cot 11^0$.

(11) Prove that $\sec 36^0 = \operatorname{cosec} 54^0$.

(12) If A be less than 90^0, prove

(i) $\cos A = \sin(90^0 - A)$.
(ii) $\tan A = \cot(90^0 - A)$.
(iii) $\sec A = \operatorname{cosec}(90^0 - A)$.
(iv) $\cot A = \tan(90^0 - A)$.

ON THE SOLUTION OF TRIGONOMETRICAL EQUATIONS.

119. A TRIGONOMETRICAL equation is an equation in which there is a letter, such as θ, which stands for an *angle* of unknown magnitude.

The *solution* of the equation is the process of finding an angle which, if it be substituted for θ, satisfies the equation.

Example 1. *Solve* $\cos\theta = \frac{1}{2}$.

This is a Trigonometrical equation. To solve it we must find some angle such that its cosine is $\frac{1}{2}$.

We know that $\cos 60^0 = \frac{1}{2}$.

Therefore if 60^0 be put for θ the equation is satisfied,

$$\therefore\ \theta = 60^0 \text{ is a solution of the equation.}$$

Example 2. *Solve* $\sin\theta - \operatorname{cosec}\theta + \frac{3}{2} = 0$.

The usual method of solution is to express all the Trigonometrical Ratios in terms of one of them.

Thus we put $\frac{1}{\sin\theta}$ for $\operatorname{cosec}\theta$, and we get

$$\sin\theta - \frac{1}{\sin\theta} + \frac{3}{2} = 0.$$

This is an equation in which θ, and therefore *sin* θ is unknown. It will be convenient if we put x for *sin* θ, and then solve the equation for x as an ordinary algebraical equation. Thus we get

$$x - \frac{1}{x} + \frac{3}{2} = 0,$$

or,

$$x^2 + \frac{3x}{2} = 1,$$

$$\therefore\ x^2 + \frac{3x}{2} + \frac{9}{16} = 1 + \frac{9}{16} = \frac{25}{16}.$$

$$\therefore\ x + \frac{3}{4} = \pm\frac{5}{4}.$$

Whence

$$x = -2, \text{ or } x = \frac{1}{2}.$$

But x stands for $\sin\theta$.

Thus we get

$$\sin\theta = -2, \text{ or } \sin\theta = \frac{1}{2}.$$

The value -2 is inadmissible for $\sin\theta$, for there is no angle whose sine is numerically greater than 1 (Art. 115).

$$\therefore\ \sin\theta = \frac{1}{2}.$$

But

$$\sin 30^0 = \frac{1}{2}.$$

$$\therefore\ \sin\theta = \sin 30^0.$$

Therefore *one* angle which satisfies this equation for θ is 30^0.

Example 3. To solve the equation $\operatorname{cosec}\theta-\cot^2\theta+1=0$.

We have $1+\cot^2\theta=\operatorname{cosec}^2\theta$. [Art. 108]

$$\therefore\ \cot^2\theta=\operatorname{cosec}^2\theta-1.$$

and we get $\operatorname{cosec}\theta-\operatorname{cosec}^2\theta+2=0$.

Let x stand for $\operatorname{cosec}\theta$. Then

$$x-x^2+2=0.$$

$$\therefore\ x^2-x=+2.$$

or $\therefore\ x^2-x+\frac{1}{4}=+\frac{1}{4}+2=\frac{9}{4}$,

$$\therefore\ x-\frac{1}{2}=\pm\frac{3}{2}.$$

Whence $x=2$, or $x=-1$,

$$\therefore\ \operatorname{cosec}\theta=2,$$

but $\operatorname{cosec}30^\circ=2$,

$$\therefore\ \operatorname{cosec}\theta=\operatorname{cosec}30^\circ.$$

therefore 30° is *one* angle which satisfies the equation for θ.

EXAMPLES. XX.

Find one angle which satisfies each of the following equations.

(1) $\sin\theta=\frac{1}{\sqrt{2}}$.

(2) $4\sin\theta=\operatorname{cosec}\theta$.

(3) $2\cos\theta=\sec\theta$.

(4) $4\sin\theta-3\operatorname{cosec}\theta=0$.

(5) $4\cos\theta-3\sec\theta=0$.

(6) $3\tan\theta=\cot\theta$.

(7) $3\sin\theta-2\cos^2\theta=0$.

(8) $\sqrt{2}\sin\theta=\tan\theta$.

(9) $2\cos\theta=\sqrt{3}\cot\theta$.

(10) $\tan\theta=3\cot\theta$.

(11) $\tan\theta+3\cot\theta=4$.

(12) $\tan\theta+\cot\theta=2$.

(13) $2\sin^2\theta+\sqrt{2}\cos\theta=2$.

(14) $4\sin^2\theta+2\sin\theta=1$.

(15) $3\tan^2\theta-4\sin^2\theta=1$.

(16) $2\sin^2\theta+\sqrt{2}\sin\theta=2$.

(17) $\cos^2\theta-\sqrt{3}\cos\theta+\frac{3}{4}=0$.

(18) $\cos^2\theta+2\sin^2\theta-\frac{5}{2}\sin\theta=0$.

* * MISCELLANEOUS EXAMPLES. XXI.

(1) Prove that $3 \sin 60^0 - 4 \sin^3 60^0 = 4 \cos^3 30^0 - 3 \cos 30^0$.

(2) Prove that $\tan 30^0 (1 + \cos 30^0 + \cos 60^0) = \sin 30^0 + \sin 60^0$.

(3) If $2 \cos^2 \theta - 7 \cos \theta + 3 = 0$, show there is only one value of $\cos \theta$.

(4) Find $\cos \theta$ from the equation $8 \cos^2 \theta - 8 \cos \theta + 1 = 0$.

(5) Find $\sin \theta$ from the equation $8 \sin^2 \theta - 10 \sin \theta + 3 = 0$, and prove that one value of θ is $\frac{\pi}{6}$.

(6) Find $\tan \theta$ from the equation $12 \tan^2 \theta - 13 \tan \theta + 3 = 0$.

(7) If $3 \cos^2 \theta + 2 . \sqrt{3} . \cos \theta = 5\frac{1}{4}$, show that there is only one value of $\cos \theta$, and that one value of θ is $\frac{\pi}{6}$.

(8) Prove that the value of $\sin^4 \theta + \cos^4 \theta + 2 . \sin^2 \theta . \cos^2 \theta$ is always the same.

(9) Simplify $\cos^4 A + 2 . \sin^2 A . \cos^2 A$.

(10) Express $\sin^6 A + \cos^6 A$ in terms of $\sin^2 A$ and powers of $\sin^2 A$.

(11) Express $1 + \tan^4 \theta$ in terms of $\cos \theta$ and its powers.

(12) Prove that $\dfrac{\cos A + \cos B}{\sin A - \sin B} + \dfrac{\sin A + \sin B}{\cos A - \cos B} = 0$.

(13) Express $(\sec A - \tan A)^2$ in terms of $\sin A$.

(14) Trace the changes in the magnitude of $\operatorname{cosec} \theta$ as θ increases from 0 to $\frac{\pi}{2}$.

(15) Trace the changes in the magnitude of $\cot \theta$ as θ decreases from $\frac{\pi}{2}$ to 0.

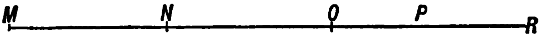
(16) Solve $\quad \sin (\theta + \phi) = \dfrac{\sqrt{3}}{2}, \cos (\theta - \phi) = \dfrac{\sqrt{3}}{2}$.

CHAPTER VIII.

On the Use of the Signs + and –.

120. The student is probably aware that, in the application of Algebra to Problems concerning *distance*, we sometimes find that the solution of an equation gives the measure of a *distance* with the sign – before it.

Example. Let M, N, O be places in a straight line; let the distance from M to N be 3 miles, and the distance from N to O, 3 miles.



One man A starting from M, rides towards O at the rate of 10 miles an hour, while another man B starting simultaneously from N, walks towards O at the rate of 4 miles an hour;

If Q be the point at which they meet, *how far is Q beyond O*?

Let P be any point beyond O, and let x be the number of miles in OP. We wish to find x, i.e. the measure of OP, so that P may coincide with Q, the point at which A overtakes B.

When A arrives at P, he has ridden $6+x$ miles. The time occupied at the rate of 10 miles an hour is $\frac{6+x}{10}$ hours.

When B arrives at P, he has walked $3+x$ miles. The time occupied at the rate of 4 miles an hour is $\frac{3+x}{4}$ hours.

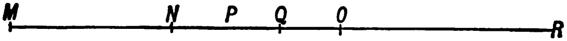
When P is the point at which they meet, these times are equal, so that

$$\frac{6+x}{10}=\frac{3+x}{4}; \quad \text{whence } x=-1.$$

Thus the required *number of miles* has the sign – before it; and we have failed to find a point *beyond* O at which A overtakes B.

121. Such a result can generally be *interpreted* by altering the statement of the problem, thus:

The line whose measure appears with the sign – before it, must be considered as drawn in the **direction opposite** to that in which it was drawn in the first statement of the problem.



Example. Taking the former example, let us alter the question as follows:

If Q be the point at which A overtakes B, how far is Q **to the left** of O?

Let P be any point *to the left* of O, and let x be the number of miles in OP.

We wish to find x (i.e. the measure of OP), so that P may coincide with Q, the point at which A overtakes B.

When A arrives at P, he has ridden $6-x$ miles.

When B arrives at P, he has walked $3-x$ miles.

Proceeding as before, we get

$$\frac{6-x}{10}=\frac{3-x}{4}; \quad \text{whence } x=+1.$$

Therefore if P is to coincide with Q (the point at which A overtakes B), OP must be one mile *to the left* of O.

122. The consideration of such examples as the above has suggested, that the sign − may be made use of, in the application of Algebra to Geometry, to represent a **direction** exactly opposite to that represented by the sign +.

Accordingly the following Rule, or Convention, has been made.

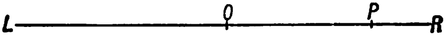
RULE. Any straight line *AB* being given, then

lines drawn parallel to *AB* in **one direction** shall be **positive**; that is, shall be *represented algebraically* by their *measures with the sign + before them:*

lines drawn parallel to *BA* in the **opposite direction** shall be **negative**; that is, shall be *represented algebraically* by their *measures with the sign − before them.*

123. We may choose for the positive direction in each case that direction which is most convenient.

Example. Let *LR* be a straight line parallel to the printed lines in the page,



and let lines drawn in the direction from *L* to *R* in the figure that is, from the left-hand towards the right, be considered *positive.* Then by the above rule, lines drawn in the direction from *R* to *L*, that is, from right to left, must be *negative.*

124. In naming a line by the letters at its extremities, we can indicate by the **order of the letters**, the direction in which the line is supposed to be drawn.

Example. Let *O* and *P* be two points in the line *LR* as in the figure, and let the measure of the distance between them be *a.*

Then *OP*, i.e. the line drawn from *O* to *P*, which is in the *positive* direction, is *represented algebraically* by $+a$.

While *PO*, i.e. the line drawn from *P* to *O*, which is in the *negative* direction, is *represented algebraically* by $-a$.

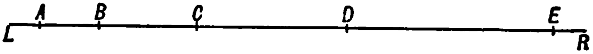
125. Hence in using the two letters at its extremities to represent a line, the student will find it advantageous always to pay careful attention to the **order of the letters.**

Example. Let LR be a straight line parallel to the printed lines in the page.

Let A, B, C, D, E be points in LR, such that the measures of AB, BC, CD, DE, are 1, 2, 3, 4 respectively.

Find the algebraical representation of

(i) $AC+CB$ (ii) $AD+DC-BC$.



(i) The algebraical representation of AC is $+3$,

the algebraical representation of CB is -2.

Hence that of $AC+CB$ is $+3-2$; that is, $+1$*.

(ii) The algebraical representation of AD is $+6$, that of DC is -3, and that of BC is $+2$.

Therefore that of $AD+DC-BC=6-3-2=+1$.

This is equivalent to that of AB.

EXAMPLES. XXII.

In the above figure, find the algebraical representation of

(1) $AB+BC+CD$.
(2) $AB+BC+CA$.
(3) $BC+CD+DE+EC$.
(4) $AD-CD$.
(5) $AD+DB+BE$.
(6) $BC-AC+AD-BD$.
(7) $CD+DB+BE$.
(8) $CD-BD+BA+AC+CE$.

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* By $AC+CB$ (attention being paid to *direction*), we mean 'Go from A to C and from C to B.' The result is equivalent to starting from A and stopping at B, *i.e.* equivalent to AB.

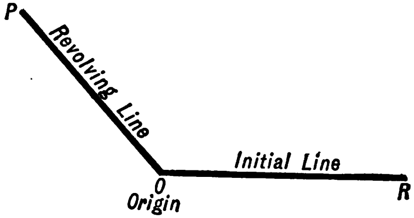
CHAPTER IX.

ON THE USE OF THE SIGNS + AND − IN TRIGONOMETRY.

126. IN **Trigonometry** in order conveniently to treat of angles of *any* magnitude, we proceed as follows.

We take a fixed point O, called the **origin**; and a fixed straight line OR, called the **initial line.**

The angle of which we wish to treat is described by a line OP, called the **revolving line.** This line OP *starts from* the initial line OR, and turns about O through an angle ROP of any proposed magnitude into the position OP.



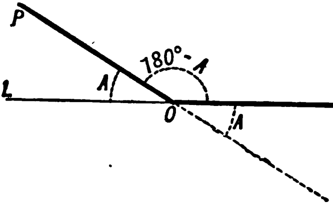
127. We have already said in Art. 41

(i) that, when an angle ROP is described by OP turning about O in the direction *contrary to that of the hands of a watch*, the angle ROP is said to be **positive**; that is, *is represented algebraically* by its *measure with the sign + before it.*

(ii) that, when an angle ROP is described by OP turning about O in the *same* direction as the hands of a watch, the angle is said to be **negative**; that is, is *represented algebraically* by its *measure with the sign − before it.*

Example. $(180^\circ - A)$ indicates

(i) the angle described by OP turning about O from the position OR in the positive direction until it has described an angle of $(180 - A)$ degrees.



Or, (ii) the angle described by OP turning about O, from the position OR, in the positive direction until it has described an angle of 180° (when it has turned into the position OL), and then turning back from OL in the negative direction through the angle $-A$ into the position OP.

Or, (iii) the angle described by OP turning about O from the position OR, in the negative direction through the angle $-A$, and then turning back in the positive direction through the angle 180°, into the position OP.

The student should observe that in each of these three ways of regarding the angle $(180^\circ - A)$, the *resulting* angle ROP is the same.

EXAMPLES. XXIII.

Draw a figure giving the position of the revolving line after it has turned through each of the following angles.

(1) 270°. (2) 370°. (3) 425°. (4) 590°.

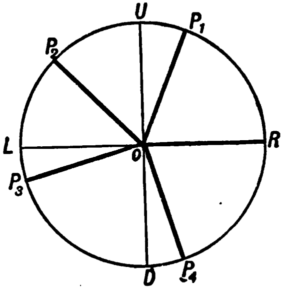
(5) -30°. (6) -330°. (7) -480°. (8) -750°.

(9) $\frac{27\pi}{4}$. (10) $2n\pi + \frac{\pi}{6}$. (11) $(2n+1)\pi + \frac{\pi}{3}$.

(12) $(2n+1)\pi - \frac{\pi}{4}$. (13) $2n\pi - \frac{\pi}{2}$. (14) $(2n+1)\pi - \frac{\pi}{2}$.

Note. $n\pi$ always stands for a whole number of two right angles.

128. It is often convenient to keep the revolving line of the same length.



In this case the point P lies always on the circumference of a circle whose centre is O.

Let this circle cut the lines LOR, UOD in the points L, R, U, D respectively.

The circle $RULD$ is thus divided at the points R, U, L, D into four **Quadrants**, of which

RU is called the **first Quadrant.**

UL is called the **second Quadrant.**

LD is called the **third Quadrant.**

DR is called the **fourth Quadrant.**

Hence, in the figure,

ROP_1 is an angle in the first Quadrant.
ROP_2 ,, ,, second Quadrant.
ROP_3 ,, ,, third Quadrant.
ROP_4 ,, ,, fourth Quadrant.

129. When we are told that an angle is in some particular Quadrant, say the third, we know that the position in which the revolving line *stops* is in the third Quadrant. But there is an unlimited number of *angles* having this same final position of OP.

Example. 25^0; 385^0 i.e. 360^0+25^0; 745^0 i.e. $2\times 360^0+25^0$; -335 i.e. -360^0+25^0 are each an angle in the first Quadrant, and are all represented *geometrically* by the *same* final position of OP.

130. Let A be an angle between $0°$ and $90°$, and let n be any *whole* number, positive or negative.

Then

(i) $2n\times 180°+A$ represents algebraically an angle in the *first* Quadrant.

(ii) $2n\times 180°-A$ represents algebraically an angle in the *fourth* Quadrant.

[For $2n\times 180^0$ represents some number n of *complete* revolutions of OP; so that after describing $n\times 360^0$, OP is again in the position OR.]

(iii) $(2n+1)\times 180^0-A$ represents algebraically an angle in the *second* Quadrant.

(iv) $(2n+1)\times 180°+A$ represents algebraically an angle in the *third* Quadrant.

[For after describing $(2n+1)\times 180^0$, OP is in the position OL.]

The corresponding expressions in circular measure are

(i) $2n\pi+\theta$; (ii) $2n\pi-\theta$; (iii) $(2n+1)\pi-\theta$;

(iv) $(2n+1)\pi+\theta$.

EXAMPLES. XXIV.

State in which Quadrant the revolving line will be after describing the following angles :

(1) 120^0. (2) 340^0. (3) 490^0.

(4) -100^0. (5) -380^0. (6) -1000^0.

(7) $\frac{2\pi}{3}$. (8) $10\pi+\frac{\pi}{4}$. (9) $9\pi-\frac{3\pi}{4}$.

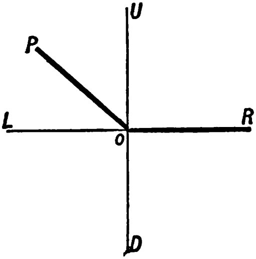
(10) $2n\pi-\frac{\pi}{4}$. (11) $(2n+1)\pi+\frac{2\pi}{3}$. (12) $n\pi+\frac{\pi}{6}$.

131. The principal **directions of lines** with which we are concerned in **Trigonometry** are as follows ;

i. that parallel to the initial line *OR* (*OR* is usually drawn from *O* towards the right hand, parallel to the printed lines in the page; and *RO* is produced to *L*.)

ii. that parallel to the line *DOU*, which is drawn through *O* at right angles to *LOR*;

iii. that parallel to the revolving line *OP*.



Accordingly we make the following **rules:**

I. Any line drawn parallel to LR in the direction **from left to right** is to be **positive**; and *consequently* (Art. 112) any line drawn parallel to RL in the *opposite* direction, i.e. from **right to left**, is to be **negative.**

II. Any line drawn parallel to DU in the direction from D to U, **upwards**, is to be **positive**; and consequently any line drawn parallel to UD in the *opposite* direction, i.e. **downwards**, is to be **negative.**

III. Any line drawn parallel to the revolving line in the direction from O to P is to be **positive**, and consequently any line drawn in the direction from P to O is to be **negative.**

Note. The student must notice that the revolving line *OP carries its positive direction round with it*, so that the line 'OP' is *always positive.*

132. We said, in Art. 81, that **the definitions of the Trigonometrical Ratios** (on pp. 46, 47), **apply to angles of any magnitude.** We have only to remark that it is generally convenient to take P on the revolving line; that PM is drawn perpendicular to the other line *produced if necessary;* and that the **order of the letters** in MP, OP, OM is an essential part of the definition.

The **order of the letters** P, M, O in the expressions $\frac{MP}{OP}$, etc., is therefore of great importance.

133. We proceed to show that the Trigonometrical Ratios of an angle vary in **Sign** according to the **Quadrant** in which the angle happens to be.

From the definition we have, with the usual letters,

$$\sin ROP = \frac{MP}{OP}, \cos ROP = \frac{OM}{OP}, \tan ROP = \frac{MP}{OM}.$$

Fig. I.

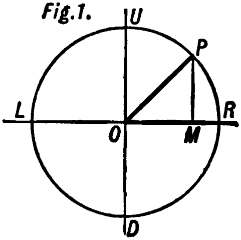


Fig. II.

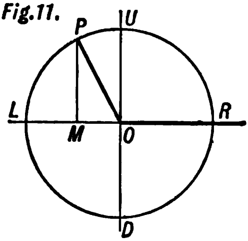


Fig. III.

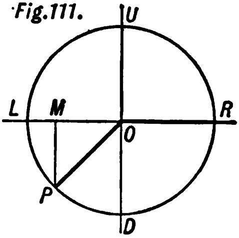
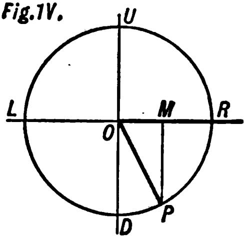


Fig. IV.



I. When *ROP* lies in the **first** Quadrant (Fig. I.).

MP is *positive* because from *M* to *P* is *upwards*
(Rule II. p. 95.)

OM is *positive* because from *O* to *M* is *towards the right*
(Rule I.)

OP is *positive* (Rule III.)

Hence, if A be any angle in the *first* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{positive};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{positive}.$$

II. When ROP lies in the **second** Quadrant, (Fig. II.).

MP is *positive*, because from M to P is *upwards*,
OM is *negative*, because from O to M is *towards the left*.
OP is *positive*.

Hence, if A be any angle in the *second* Quadrant,

$$\sin A, \text{ which is } \frac{MP}{OP}, \text{ is } \textit{positive};$$

$$\cos A, \text{ which is } \frac{OM}{OP}, \text{ is } \textit{negative};$$

$$\tan A, \text{ which is } \frac{MP}{OM}, \text{ is } \textit{negative}.$$

III. When ROP lies in the **third** Quadrant (Fig. III.) MP is *negative*, OM is *negative*, OP is *positive*.

So that, if A be any angle in the *third* Quadrant,
sin A is *negative*, cos A is *negative*, tan A is *positive*.

IV. When ROP lies in the **fourth** Quadrant (Fig. IV) MP is *negative*, OM is *positive*, OP is *positive*.

So that, if A be any angle in the *fourth* Quadrant,
sin A is *negative*, cos A is *positive*, tan A is *negative*.

134. The table given below exhibits the results of the last article.

Quadrant ...	I.	II.	III.	IV.
Sine	+	+	−	−
Cosine	+	−	−	+
Tangent ...	+	−	+	−

The student should notice that in any particular Quadrant the three *signs* of sine, cosine, and tangent are unlike their signs in any other Quadrant.

EXAMPLES. XXV.

State the *sign* of the sine, cosine, and tangent of each of the following angles :

(1) 60^0. (2) 135^0. (3) 265^0.

(4) 275^0. (5) -10^0. (6) -91^0.

(7) -193^0. (8) -350^0. (9) -1000^0.

(10) $2n\pi + \frac{\pi}{4}$. (11) $2n\pi + \frac{3\pi}{4}$. (12) $2n\pi - \frac{\pi}{6}$.

135. The NUMERICAL VALUES through which the Trigonometrical Ratios pass, as the angle *ROP* moves through the *first* Quadrant, are **repeated** as the angle *ROP* moves through each of the other Quadrants.

Thus as *P* moves through the second Quadrant from *U* to *L*, Fig. II, (*OP* being always of the same length) *MP* and *OM* pass through the same succession of numerical values through which they pass, as P moves through the first Quadrant in the *opposite* direction from U to R.

Example 1. *Find the sine, cosine and tangent of* 120^0.

120^0 is an angle in the second Quadrant.

Let the angle ROP be 120^0 (Fig. II. p. 96.)

Then the angle $POL = 180^0 - 120^0 = 60^0$.

Hence, $\sin 120^0 = \frac{MP}{OP} = \sin 60^0$ *numerically*, and in the *second* Quadrant the sine is *positive*.

Therefore $\sin 120^0 = \frac{\sqrt{3}}{2}$ ....(i).

Again, $\cos 120^0 = \frac{OM}{OP} = \cos 60^0$, *numerically*, and in the *second* Quadrant the cosine is *negative*.

Therefore $\cos 120^0 = -\frac{1}{2}$ ....(ii).

Similarly, $\tan 120^0 = -\sqrt{3}$ ....(iii).

Example 2. *Find the sine, cosine and tangent of* 225^0.

225^0 is an angle in the third Quadrant.

Let the angle ROP be 225^0 (Fig. III.)

Here the angle $POL = 225^0 - 180^0 = 45^0$.

Therefore the Trigonometrical Ratios of $225^0 =$ those of 45^0 *numerically*; and in the *third* Quadrant the sine and cosine are each *negative* and the tangent is *positive*.

Hence, $\sin 225^0 = -\frac{1}{\sqrt{2}}$; $\cos 225^0 = -\frac{1}{\sqrt{2}}$; $\tan 225^0 = 1$.

136. The cosecant, secant and cotangent of an angle A have the same *sign* as the sine, cosine, and tangent of A respectively.

EXAMPLES. XXVI.

Find the algebraical value of the sine, cosine and tangent of the following angles :

(1) 150^0. (2) 135^0. (3) -240^0. (4) 330^0.
(5) -45^0. (6) -300^0. (7) 225^0. (8) -135^0.
(9) 390^0. (10) 750^0. (11) -840^0. (12) 1020^0.
(13) $2n\pi+\frac{\pi}{4}$. (14) $(2n+1)\pi-\frac{\pi}{3}$. (15) $(2n-1)\pi+\frac{\pi}{6}$.

* *Example. To trace the changes in the magnitude and sign of sin* A, *as* A *increases from* 0^0 *to* 360^0.

Take the figure and construction of page 96.

As A increases from 0^0 to 90^0, MP increases from zero to OP, and is positive.

Therefore sin A increases from 0 to 1 and is positive.

As A increases from 90^0 to 180^0, MP decreases from OP to zero, and is positive.

Therefore sin A decreases from 1 to 0 and is positive.

As A increases from 180^0 to 270^0, MP increases from zero to OP, and is negative.

Therefore sin A increases numerically from 0 to 1 and is negative.

As A increases from 270^0 to 360^0, MP decreases from OP to zero, and is negative.

Therefore sin A decreases numerically from 1 to 0 and is negative.

* EXAMPLES. XXVII.

Trace the changes in sign and magnitude as A increases from 0^0 to 360^0 of

(1) $\cos A$. (2) $\tan A$. (3) $\cot A$. (4) $\sec A$.
(5) $\operatorname{cosec} A$. (6) $1-\sin A$. (7) $\sin^2 A$. (8) $\sin A . \cos A$.
(9) $\sin A+\cos A$. (10) $\tan A+\cot A$. (11) $\sin A-\cos A$.

CHAPTER X**.

ON ANGLES UNLIMITED IN MAGNITUDE.

137. JUST as the definitions of the Trigonometrical Ratios apply to angles of any magnitude whatever, so **every general Formula** involving these Ratios **is true for angles of any magnitude whatever.**

It is most important that the student should examine for himself into the truth of this statement.

138. The formulæ

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \cot A = \frac{1}{\tan A},$$

are really definitions; and since the definitions apply, therefore these formulæ are true, whatever be the magnitude of A.

The formulæ $\tan A = \dfrac{\sin A}{\cos A}$, $\cot A = \dfrac{\cos A}{\sin A}$,

are deduced immediately from the definitions, and therefore they are true whatever be the magnitude of A.

139. The formulæ

$$\sin^2 A + \cos^2 A = 1,$$
$$1 + \tan^2 A = \sec^2 A,$$
$$1 + \cot^2 A = \operatorname{cosec}^2 A,$$

are each a trigonometrical statement of Euc. I. 47, and depend only on the fact that MP, OM, and OP are the sides of a *right-angled* triangle. That this is the case, whatever be the magnitude of the angle A, is evident from the figures on page 96.

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* * To be omitted on a first reading, except pp. 104, 105.

140. In Art. 118 we proved that the sine of an angle is equal to the cosine of its complement, provided the angle lies between 0° and 90°.

We now give some examples of a method of proving the truth of this and other like formulæ, **whatever** be the magnitude of the angle concerned.

Example 1. *To prove that the sine of an angle = the cosine of its complement.*

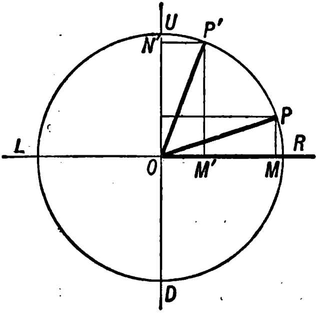
That is, *to prove* $\sin A = \cos (90^\circ - A)$

and $\cos A = \sin (90^\circ - A)$.

We take **two revolving lines** OP and OP'. OP starting from OR is to describe the angle A; OP' starting from OR is to describe $(90^\circ - A)$.

As usual, PM, $P'M'$ are perpendiculars on OR and $P'N'$ is a perpendicular on OU.

In describing $(90^\circ - A)$ we shall consider that OP' starting from OR, turns first through 90° into the position OU, and then turns back from OU through the angle $UOP' = (-A)$.



So that ROP, the angle which OP describes from OR, is always equal to UOP', the angle which OP' describes from OU in the *opposite* direction.

Hence, $N'P'$, that is OM', is *always* equal to MP in *magnitude.*

Also it will be seen that when P is *above* LOR, P' is to the *right* of UOD; when P is *below* LOR, P' is to the *left* of UOD.

Hence, OM' and MP have *always* the *same sign.*

Therefore $$\frac{MP}{OP}=\frac{OM'}{OP'} \text{ always,}$$

or, $$\sin A=\cos(90^\circ-A), \text{ for all values of } A.$$

Again, ON', that is $M'P'$, is *always* equal to OM in *magnitude.*

And P' is above or below LOR according as P is to the right or to the left of UOD.

So that $M'P'$ and OM have *always* the *same sign.*

Therefore $$\frac{OM}{OP}=\frac{M'P'}{OP'} \text{ always,}$$

or, $$\cos A=\sin(90^\circ-A) \text{ for all values of } A.$$

EXAMPLES. XXVIII.

Prove, drawing a separate figure for each example, that

(1) $\sin 30^\circ=\cos 60^\circ$.

(2) $\sin 65^\circ=\cos 25^\circ$.

(3) $\sin 195^\circ=\cos(-105^\circ)$.

(4) $\cos 275=\sin(-185^\circ)$.

(5) $\cos(-27^\circ)=\sin 117^\circ$.

(6) $\cos 300^\circ=\sin(-210^\circ)$.

If A, B, C be the angles of a triangle, so that $A+B+C=180^\circ$, prove

(7) $\cos\frac{A}{2}=\sin\frac{B+C}{2}$.

(8) $\cos\frac{B}{2}=\sin\frac{A+C}{2}$.

(9) $\sin\frac{C}{2}=\cos\frac{A+B}{2}$.

(10) $\sin\frac{A}{2}=\cos\frac{B+C}{2}$.

141. **Def.** Two angles are said to be the **Supplements** the one of the other when their sum is two right angles.

Thus $(180^0 - A)$ is the **supplement** of A

If A, B, C be the angles of a triangle, $(A + B + C) = 180^0$, so that $(B + C)$ is the **supplement** of A.

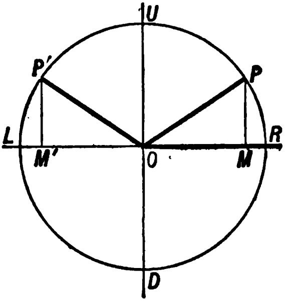
Example 2. *To prove that the sine of an angle = the sine of its supplement; and that the cosine of an angle* = – (*the cosine of its supplement*).

That is, *to prove* $\sin A = \sin(180^0 - A)$

and $$\cos A = -\cos(180^0 - A).$$

We take **two revolving lines** OP and OP'. OP starting from OR describes the angle A; OP' starting from OR describes the angle $(180^0 - A)$.

In describing $(180^0 - A)$ we consider that OP' starting from OR turns first through 180^0 into the position OL, and then back from OL through the angle $LOP' = (-A)$.



So that ROP, the angle which OP describes from OR, is always equal to LOP', the angle which OP' describes from OL in the *opposite* direction.

Hence, MP and $M'P'$ are *always* equal in *magnitude.*

Also, P and P' are always *both* above, or *both* below LOR.

So that MP and $M'P'$ are *always* of the *same sign.*

Therefore $$\frac{MP}{OP}=\frac{M'P'}{OP'} \text{ always,}$$

or, $$\sin A=\sin(180^0-A), \text{ for all values of } A.$$

Again, OM and OM' are *always* equal in *magnitude.*

Also it will be seen that when P is on the *right* of UOD, P' is on the *left* of UOD; when P is on the *left* of UOD, P' is on *right* of UOD.

So that OM and OM' are *always* of *opposite sign.*

Therefore $$\frac{OM}{OP}=-\frac{OM'}{OP'} \text{ always,}$$

or, $$\cos A=-\cos(180^0-A), \text{ for all values of } A.$$

EXAMPLES. XXIX.

Prove, drawing a separate figure in each case, that

(1) $\sin 60^0=\sin 120^0$. (2) $\sin 340^0=\sin(-160^0)$.

(3) $\sin(-40^0)=\sin 220^0$. (4) $\cos 320^0=-\cos(-140^0)$.

(5) $\cos(-380^0)=-\cos 560^0$. (6) $\cos 195^0=-\cos(-15^0)$.

If A, B, C be the angles of a triangle, prove

(1) $\sin A=\sin(B+C)$. (2) $\sin C=\sin(A+B)$.

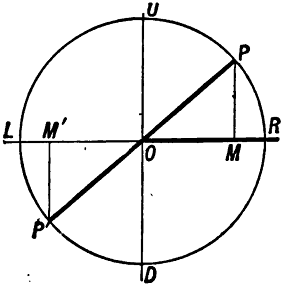
(3) $\cos B=-\cos(A+C)$. (4) $\cos A=-\cos(C+B)$.

Example 3. *To prove* $\sin A = -\sin(180^\circ + A)$,

and $\cos A = -\cos(180^\circ + A)$.

As before, we take **two revolving lines** OP and OP'. OP starting from OR describes the angle A; OP' starting from OR describes the angle $(180^\circ + A)$.

In describing $(180^\circ + A)$ we consider that OP' starting from OR turns first through 180° into the position OL, and then on from OL through the angle A.



So that ROP, the angle which OP describes from OR, is always equal to LOP', the angle which OP' describes from OL in the *same* direction.

Hence, *$M'P'$ always $= MP$* in *magnitude.*

Also it will be seen that when P is *above* LOR, P' is *below* LOR; and *vice versâ**.

So that MP and $M'P'$ are *always* of *opposite sign*

Hence, $$\frac{MP}{OP} = -\frac{M'P'}{OP'} \text{ always,}$$

or, $$\sin A = -\sin(180^\circ + A), \text{ for all values of } A.$$

Similarly, *OM always $= OM'$* in *magnitude.*

And P' is to the left or to the right of UOD according as P is to the right or to the left of UOD.

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* This will be more clear if the student observes that POP' is always a straight line.

Hence, $$\frac{OM}{OP} = -\frac{OM'}{OP'} \text{ always,}$$

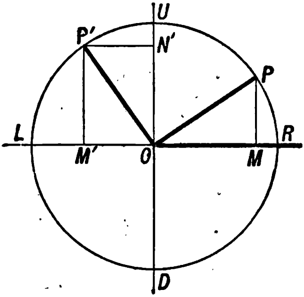
or, $$\cos A = -\cos(180^0 + A), \text{ for all values of } A.$$

Example 4. *To prove* $\sin A = -\cos(90^0 + A)$,

and $$\cos A = \sin(90^0 + A).$$

As before, we take **two revolving lines** OP and OP'. OP starting from OR describes the angle A; OP' starting from OR describes the angle $90^0 + A$.

In describing $(90^0 + A)$ we consider that OP' starting from OR turns first through 90^0 into the position OU, and then on from OU through the angle A.



So that ROP, the angle which OP describes from OR, is always equal to UOP', the angle which OP' describes from OU in the *same* direction.

Hence, $N'P'$, that is OM', *always* $= MP$ in *magnitude.*

Also, P' is to the left or to the right of UOD according as P is above or below LOR.

So that MP and OM' are always of *opposite sign.*

Hence, $$\frac{MP}{OP} = -\frac{OM'}{OP'} \text{ always,}$$

or, $$\sin A = -\cos(90^0 + A), \text{ for all values of } A.$$

Similarly, *OM always = M'P' in magnitude.*

And P' is above or below LOR according as P is to the right or to the left of UOD.

So that OM and $M'P'$ are *always* of the *same sign.*

Hence, $$\frac{OM}{OP}=\frac{M'P'}{OP'} \text{ always,}$$

or, $$\cos A=\sin(90^0+A), \text{ for all values of } A.$$

MISCELLANEOUS EXAMPLES. XXX.

Prove, drawing a separate figure in each case, that

(1) $\sin 60^0=-\sin 240^0$.

(2) $\sin 170^0=-\sin 350^0$.

(3) $\sin(-20^0)=-\sin 160^0$.

(4) $\cos 380^0=-\cos 560^0$.

(5) $\cos(-225)=-\cos(-45^0)$.

(6) $\cos 1005^0=-\cos 1185^0$

(7) $\sin 60^0=-\cos 150^0$.

(8) $\cos 60^0=\sin 150^0$

(9) $\sin 225=-\cos 315^0$.

(10) $\cos(-60^0)=\sin 30^0$.

If $A+B+C$ be the angles of a triangle, prove that

(11) $\sin A=-\sin(2A+B+C)$.

(12) $\sin A=-\cos\dfrac{3A+B+C}{2}$

(13) $\cos B=\sin\dfrac{A+3B+C}{2}$

(14) $\cos C=-\cos(A+B+2C)$.

(15) $\cos\dfrac{B-C}{2}=\sin\dfrac{A+2B}{2}$

(16) $\sin\dfrac{C-A}{2}=-\cos\dfrac{B+2C}{2}$

Prove the following statements for all values of A and of a.

(17) $\sin A = -\sin(-A)$. (18) $\cos A = \cos(-A)$.

(19) $\sin A = \cos(A - 90^0)$. (20) $\cos A = -\sin(A - 90^0)$.

(21) $\sin a = \cos\left(\frac{3\pi}{2} + a\right)$.

(22) $\cos a = -\sin\left(\frac{3\pi}{2} + a\right)$

(23) $\sin a = -\cos\left(\frac{3\pi}{2} - a\right)$.

(24) $\cos a = -\sin\left(\frac{3\pi}{2} - a\right)$.

(25) $\sin\left(\frac{\pi}{2} - a\right) = \sin\left(\frac{\pi}{2} + a\right)$.

(26) $\cos(\pi + a) = \cos(\pi - a)$.

(27) $\tan(90^0 - A) = \cot A$. (28) $\tan A = -\tan(-A)$.

(29) $\tan(90^0 + A) = -\cot(A)$. (30) $\tan a = -\tan(\pi - a)$.

(31) $\tan A = \tan(180^0 + A)$. (32) $\cot\left(\frac{\pi}{2} - a\right) = \tan a$.

142. We have seen (Art. 135) that there are many angles of different magnitude which have the same *sine*.

If two such angles are in the *same Quadrant* they are represented geometrically by the same position of OP, so that they differ by some multiple of four right angles.

143. If we are given the value of the sine of an angle it is important to be able to find, Geometrically and Algebraically, all angles which have that value for their sine.

144. *To find the complete Geometrical Solution of the equation* **sin** $\theta = a$.

With the usual construction, let the radius of the circle $RULD$ be the unit of length; then the measure of OP is 1.

Fig.1.

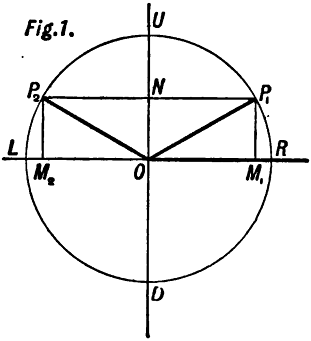
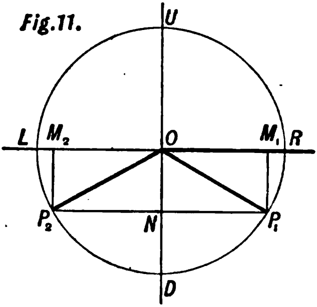


Fig.11.



From O draw on OU a line ON so that its measure is a.

[ON will be drawn upwards (Fig. I.) or downwards (Fig. II.) from O according as a is positive or negative.]

Through N draw P_1P_2 parallel to LOR to cut the circle in P_1 and P_2*. Join OP_1, OP_2. Draw P_1M_1, P_2M_2 perpendicular to LOR.

Then $M_1P_1 = ON = M_2P_2$, and $\dfrac{M_1P_1}{OP_1} = \dfrac{a}{1} = \dfrac{M_2P_2}{OP_2}$.

Hence any angle described by the revolving line OP, when OP, starting from OR, stops either in the position OP_1, or in the position OP_2, is an angle whose **sine** is a.

And no other angle has its sine equal to a, for there is no other possible position of N.

145. *To find the complete Algebraical Solution of the equation* **sin** $\theta = a$.

---

* If a were *greater than unity*, ON would be greater than OU, and the construction would fail.

With the figure and construction of the last article; let α be the least angle whose sine $= a$.

Then in the figure $ROP_1 = \alpha$ and $P_2OL = \alpha$.

Every angle whose sine $= a$ is included among those described by the revolving line OP when, starting from OR, OP stops in one or the other of the positions OP_1 or OP_2.

If OP, starting from OR, stop in the position OP_1, the angle described is one of those included in the expression

$$2m\pi + \alpha,$$

where m is some integer, positive or negative. [Art. 130.]

If OP, starting from OR, stop in the position OP_2, the angle described is one of those included in the expression

$$(2r + 1)\,\pi - \alpha,$$

where r is some integer, positive or negative*.

Both of these expressions are included in

$$n\pi + (-1)^n\,\alpha,$$

where n is some integer, positive or negative*.

Thus the solution of the equation $\sin\theta = \sin\alpha$ is

$$\theta = n\pi + (-1)^n\,\alpha.$$

Example. Find six angles between -4 right angles and $+8$ right angles which satisfy the equation $\sin A = \sin 18^0$.

We have $\theta = n\pi + (-1)^n \frac{\pi}{10}$, or $A = n \times 180^0 + (-1)^n\, 18^0$.

Put for n the values $-2, -1, 0, 1, 2, 3, 4$ successively and we get the six angles

$$-360^0 + 18^0,\ -180^0 - 18^0,\ 18^0,\ 180^0 - 18^0,\ 360^0 + 18^0,\ 540^0 - 18^0,$$

$$\text{i.e. } -342^0,\ -198^0,\ 18^0,\ 162^0,\ 378^0,\ 522^0.$$

---

* For if n be *even*, let it be $2m$, when $(-1)^{2m} = +1$; if n be *odd*, let it be $2r + 1$, when $(-1)^{2r+1} = -1$.

The student is recommended to draw a figure in the above example. Also to draw a figure in each example of this kind which he works for exercise.

EXAMPLES. XXXI.

(1) Find the four smallest angles which satisfy the equations

(i) $\sin A = \frac{1}{2}$. (ii) $\sin A = \frac{1}{\sqrt{2}}$. (iii) $\sin A = \frac{\sqrt{3}}{2}$. (iv) $\sin A = -\frac{1}{2}$.

(2) Find four angles between zero and +8 right angles which satisfy the equations

(i) $\sin A = \sin 20^\circ$. (ii) $\sin\theta = -\frac{1}{\sqrt{2}}$. (iii) $\sin\theta = -\sin\frac{\pi}{7}$.

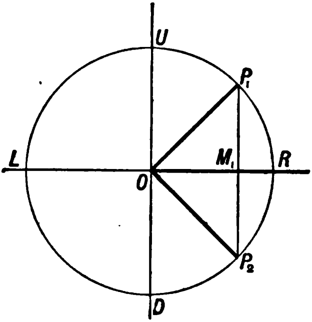
(3) Find the complete algebraical solution of

(i) $\sin\theta = -\frac{1}{2}$. (ii) $2\sin^2\theta + 3\sin\theta = 2$. (iii) $\sin^2\theta = \cos^2\theta$.

(4) Prove that 30°, 150°, -330°, 390°, -210° have all the same sine.

146. *To find the complete Geometrical Solution of the equation* $\cos\theta = a$.

With the usual construction, let the radius of the circle *RULD* be the unit of length, so that the measure of *OP* is 1.



From *O* draw on *OR* a line OM_1 such that its measure is *a*.

OM_1 will be drawn towards the right or towards the left according as *a* is positive or negative.

Through M_1 draw $P_1M_1P_2$ perpendicular to OR to cut the circle in P_1, P_2. Join OP_1, OP_2.

Then $$\frac{OM_1}{OP_1} = \frac{a}{1} = \frac{OM_1}{OP_2}.$$

Hence any angle described when OP, starting from OR, stops in the position OP_1, or in the position OP_2, is an angle whose cosine is a.

And no other angle has its cosine $= a$, for there is no other possible position of M_1.

147. *To find the complete Algebraical Solution of the equation* $\cos\theta = a$.

With the figure and construction of the last article; let α be the least angle whose cosine $= a$.

Then in the figure $ROP_1 = \alpha$ and $P_2OR = \alpha$.

Every angle whose cosine $= a$ is included among those described when OP, starting from OR, stops in one or the other of the positions OP_1 or OP_2.

If OP starting from OR stop in the position OP_1, the angle described is one of those included in the expression

$$2m\pi + \alpha. \qquad \text{[Art. 130.]}$$

If OP starting from OR stop in the position OP_2, the angle described is one of those included in the expression

$$2r\pi - \alpha.$$

Both of these expressions are included in

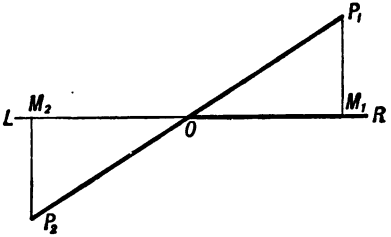
$$2n\pi \pm \alpha.$$

Thus the solution of the equation $\cos\theta = \cos\alpha$ is

$$\theta = 2n\pi \pm \alpha.$$

148. *To find the complete Geometrical Solution of the equation* **tan** $\theta = a$.

From O draw on the line OR two lines OM_1, OM_2, whose measures are $+1$ and -1 respectively.



And draw perpendicular to LOR from M_1 a line M_1P_1 whose measure is a, and from M_2 a line M_2P_2 whose measure is $-a$. Join OP_1, OP_2.

Then $\dfrac{M_1P_1}{OM_1} = \dfrac{a}{1} = a$, and $\dfrac{M_2P_2}{OM_2} = \dfrac{-a}{-1} = a$.

Hence any angle described when OP, starting from OR, stops in one or the other of the positions OP_1 or OP_2, is an angle whose **tangent** is a.

And no other angle has its tangent $= a$.

149. *To find the complete Algebraical Solution of the equation* **tan** $\theta = a$.

With the figure and construction of the last article; let α be the least angle whose tangent is a.

Then in the figure $ROP_1 = \alpha$ and $LOP_2 = \alpha$.

Every angle whose tangent is a is included among those described by OP, starting from OR and stopping in one or the other of the positions OP_1 or OP_2.

If OP starting from OR stop in the position OP_1, the angle described is one of those included in the expression

$$2m\pi + \alpha. \qquad \text{[Art. 130.]}$$

If OP starting from OR stop in the position OP_2, the angle described is one of those included in the expression

$$(2r+1)\,\pi + \alpha.$$

Both of these expressions are included in

$$n\pi + \alpha^*.$$

Thus the solution of the equation $\tan\theta = \tan\alpha$ is

$$\theta = n\pi + \alpha.$$

EXAMPLES. XXXII.

(1) Write down the complete Algebraical Solution of each of the following equations:

(i) $\cos\theta = \frac{1}{2}$. (ii) $\tan\theta = 1$. (iii) $\tan\theta = -1$.

(iv) $\tan\theta = -\sqrt{3}$. (v) $\cos\theta = \cos\frac{4\pi}{5}$. (vi) $\tan\theta = \tan\frac{3\pi}{4}$.

(2) Show that each of the following angles has the same cosine:

$$-120^0, \quad 240^0, \quad 480^0, \quad -480^0.$$

(3) The angles 60^0 and -120^0 have one of the Trigonometrical Ratios the same for both; which of the ratios is it?

(4) Can the following angles have any one of their Trigonometrical Ratios the same for all?

$$-23^0, \quad -157^0 \text{ and } 157^0.$$

(5) Find four angles which satisfy each of the equations in (1).

---

* For if n be even, this is the first formula; if n be odd it is the second.

150. We can now point out the use of the ambiguous sign ± in the formula $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

If we know the numerical value of the sine of an angle θ, without knowing the magnitude of the angle, we cannot from the identity, $\cos^2\theta = 1 - \sin^2\theta$, *completely* determine $\cos\theta$, for we get $\cos\theta = \pm\sqrt{1-\sin^2\theta}$.

This is a **general formula**, and we shall find that it represents an important Geometrical truth.

151. Given $\sin\theta = a$, we can say that θ is *one of the angles* represented by one or the other of the positions OP_1, OP_2 of the revolving line in Fig. I. on page 110.

If we attempt to find the cosine of these angles we get two different values for the cosine; for $\frac{OM_1}{OP_1}$ and $\frac{OM_2}{OP_2}$ although equal in magnitude, are *opposite in sign*. Hence, if α be the least angle whose sine is equal to a, we have

$$\cos\theta = \pm\cos\alpha = \pm\sqrt{1-\sin^2\alpha}.$$

152. The same result may be obtained from the formula $\theta = n\pi + (-1)^n\alpha$. For $\cos\{n\pi + (-1)^n\alpha\}$ is of different sign according as n is even or odd.

EXAMPLES. XXXIII.

(1) If θ be found from the equation $\cos\theta = a$, show geometrically that there are two values of $\sin\theta$ and of $\tan\theta$.

(2) If θ be found from the equation $\tan\theta = a$, show geometrically that there are two values of $\sin\theta$ and of $\cos\theta$.

(3) If A be the *least* angle without regard to sign such that $\sin A = a$, show that $\cos A = +\sqrt{1-\sin^2 A}$.

(4) If A be the least *positive* angle such that $\cos A = a$, prove that $\sin A = +\sqrt{1-\cos^2 A}$.

CHAPTER XI.

ON THE TRIGONOMETRICAL RATIOS OF TWO ANGLES.

153. WE proceed to establish the following fundamental formulæ:

$$\left.\begin{aligned}\sin(A+B)&=\sin A\,.\cos B+\cos A\,.\sin B\\ \cos(A+B)&=\cos A\,.\cos B-\sin A\,.\sin B\\ \sin(A-B)&=\sin A\,.\cos B-\cos A\,.\sin B\\ \cos(A-B)&=\cos A\,.\cos B+\sin A\,.\sin B\end{aligned}\right\}\ \dots\ \text{(i)}.$$

Here, A and B are angles; so that $(A+B)$ and $(A-B)$ are also angles.

Hence, $\sin(A+B)$ is the sine of an angle, and must not be confounded with $\sin A+\sin B$.

Sin $(A+B)$ is a single fraction.

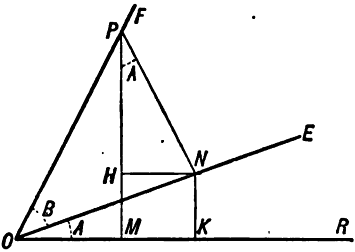
Sin $A+\sin B$ is the sum of two fractions.

154. The *proofs* given in the next two pages are perfectly general, as will be explained below (cf. Art. 169); but the *figures* are drawn for the simplest case in each.

The student should notice that the words of the two proofs are very nearly the same.

To prove that $\sin(A+B)=\sin A \,.\, \cos B+\cos A \,.\, \sin B$,

and that $\cos(A+B)=\cos A \,.\, \cos B-\sin A \,.\, \sin B$.



Let ROE be the angle A, and EOF the angle B. Then in the figure, ROF is the angle $(A+B)$.

In OF, *the line which bounds the compound angle* $(A+B)$, take any point P, and from P draw PM, PN at right angles to OR and OE respectively. Draw NH, NK at right angles to MP and OR respectively. Then the angle

$$NPH=90^0-HNP=HNO=ROE=A^*.$$

Now $\sin(A+B)=\sin ROF=\dfrac{MP}{OP}=\dfrac{MH+HP}{OP}=\dfrac{KN}{OP}+\dfrac{HP}{OP}$

$$=\frac{KN\,.\,\mathrm{ON}}{\mathrm{ON}\,.\,OP}+\frac{HP\,.\,\mathrm{NP}}{\mathrm{NP}\,.\,OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}+\frac{HP}{NP}\cdot\frac{NP}{OP}$$

$$=\sin ROE\,.\,\cos EOF+\cos HPN\,.\,\sin EOF$$

$$=\sin A\,.\,\cos B+\cos A\,.\,\sin B.$$

Also $\cos(A+B)=\cos ROF=\dfrac{OM}{OP}=\dfrac{OK-MK}{OP}=\dfrac{OK}{OP}-\dfrac{HN}{OP}$

$$=\frac{OK\,.\,\mathrm{ON}}{\mathrm{ON}\,.\,OP}-\frac{HN\,.\,\mathrm{NP}}{\mathrm{NP}\,.\,OP}=\frac{OK}{ON}\cdot\frac{ON}{OP}-\frac{HN}{NP}\cdot\frac{NP}{OP}$$

$$=\cos ROE\,.\,\cos EOF-\sin HPN\,.\,\sin EOF$$

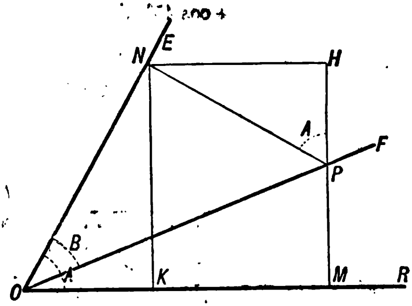
$$=\cos A\,.\,\cos B-\sin A\,.\,\sin B.$$

---

* Or thus. A circle goes round $OMNP$, because the angles OMP and ONP are right angles; therefore MPN and MON are angles in the same segment; so that $MPN=MON=A$.

To prove that $\sin(A-B)=\sin A\,.\cos B-\cos A\,.\sin B$,

and that $\cos(A-B)=\cos A\,.\cos B+\sin A\,.\sin B$.



Let ROE be the angle A, and FOE the angle B. Then in the figure, ROF is the angle $(A-B)$.

In OF, *the line which bounds the compound angle* $(A-B)$, take any point P, and from P draw PM, PN at right angles to OR and OE respectively. Draw NH, NK at right angles to MP and OR respectively. Then the angle

$$NPH=90^0-HNP=HNE=ROE=A^*.$$

Now $\sin(A-B)=\sin ROF=\dfrac{MP}{OP}=\dfrac{MH-PH}{OP}=\dfrac{KN}{OP}-\dfrac{PH}{OP}$

$$=\frac{KN\,.\,\mathrm{ON}}{\mathrm{ON}\,.\,OP}-\frac{PH\,.\,\mathrm{NP}}{\mathrm{NP}\,.\,OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}-\frac{PH}{NP}\cdot\frac{NP}{OP}$$

$$=\sin ROE\,.\cos FOE-\cos HPN\,.\sin FOE$$

$$=\sin A\,.\cos B-\cos A\,.\sin B.$$

Also $\cos(A-B)=\cos ROF=\dfrac{OM}{OP}=\dfrac{OK+KM}{OP}=\dfrac{OK}{OP}+\dfrac{NH}{OP}$

$$=\frac{OK\,.\,\mathrm{ON}}{\mathrm{ON}\,.\,OP}+\frac{NH\,.\,\mathrm{NP}}{\mathrm{NP}\,.\,OP}=\frac{OK}{ON}\cdot\frac{ON}{OP}+\frac{NH}{NP}\cdot\frac{NP}{OP}$$

$$=\cos ROE\,.\cos FOE+\sin HPN\,.\sin FOE$$

$$=\cos A\,.\cos B+\sin A\,.\sin B.$$

---

* **Or thus. A circle goes round *OMPN*, because the angles *OMP* and *ONP* are right angles; therefore *MPN* and *MON* together make up two right angles; so that *HPN*=*MON*=*A*.**

Example. Find the value of $\sin 75^0$.

$$\begin{aligned}\sin 75^0 &= \sin(45^0 + 30^0)\\ &= \sin 45^0 . \cos 30^0 + \cos 45^0 . \sin 30^0\\ &= \frac{1}{\sqrt{2}} . \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} . \frac{1}{2}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{2}(\sqrt{3}+1)}{4}.\end{aligned}$$

EXAMPLES. XXXIV.

(1) Show that $\cos 75^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(2) Show that $\sin 15^0 = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(3) Show that $\cos 15^0 = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

(4) Show that $\tan 75^0 = 2 + \sqrt{3}$.

(5) If $\sin A = \frac{4}{5}$ and $\sin B = \frac{3}{5}$, find a value for $\sin(A+B)$ and for $\cos(A-B)$.

(6) If $\sin A = {\cdot}6$ and $\sin B = \frac{5}{13}$, find a value for $\sin(A-B)$ and for $\cos(A+B)$.

(7) If $\sin A = \dfrac{1}{\sqrt{5}}$ and $\sin B = \dfrac{1}{\sqrt{10}}$, show that one value of $(A+B)$ is 45^0.

(8) Prove that $\sin 75^0 = {\cdot}9659\ldots$

(9) Prove that $\sin 15^0 = {\cdot}2588\ldots$

(10) Prove that $\tan 15^0 = {\cdot}2679\ldots$

155. It is important that the student should become thoroughly familiar with the formulæ proved on the last two pages, and that he should be able to work examples involving their use.

EXAMPLES. XXXV.

Prove the following statements.

(1) $\sin(A+B)+\sin(A-B)=2\sin A.\cos B.$

(2) $\sin(A+B)-\sin(A-B)=2\cos A.\sin B.$

(3) $\cos(A+B)+\cos(A-B)=2\cos A.\cos B.$

(4) $\cos(A-B)-\cos(A+B)=2\sin A.\sin B.$

(5) $\dfrac{\sin(A+B)+\sin(A-B)}{\cos(A+B)+\cos(A-B)}=\tan A.$

(6) $\tan\alpha+\tan\beta=\dfrac{\sin(\alpha+\beta)}{\cos\alpha.\cos\beta}.$

(7) $\tan\alpha-\tan\beta=\dfrac{\sin(\alpha-\beta)}{\cos\alpha.\cos\beta}.$

(8) $\cot\alpha+\tan\beta=\dfrac{\cos(\alpha-\beta)}{\sin\alpha.\cos\beta}.$

(9) $\cot\alpha-\tan\beta=\dfrac{\cos(\alpha+\beta)}{\sin\alpha.\cos\beta}.$

(10) $\tan\alpha+\cot\beta=\dfrac{\cos(\alpha-\beta)}{\cos\alpha.\sin\beta}.$

(11) $\dfrac{\tan\theta+\tan\phi}{\tan\theta-\tan\phi}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(12) $\dfrac{\tan\theta.\tan\phi+1}{1-\tan\theta.\tan\phi}=\dfrac{\cos(\theta-\phi)}{\cos(\theta+\phi)}.$

(13) $\dfrac{\tan\theta+\cot\phi}{\cot\phi-\tan\theta}=\cos(\theta-\phi).\sec(\theta+\phi).$

(14) $\dfrac{\cot\theta+\cot\phi}{\cot\theta-\cot\phi}=-\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(15) $\dfrac{\tan\theta.\cot\phi+1}{\tan\theta.\cot\phi-1}=\dfrac{\sin(\theta+\phi)}{\sin(\theta-\phi)}.$

(16) $\dfrac{1+\cot\gamma.\tan\delta}{\cot\gamma-\tan\delta}=\tan(\gamma+\delta).$

(17) $\dfrac{1-\cot\gamma.\tan\delta}{\cot\gamma+\tan\delta}=\tan(\gamma-\delta).$

(18) $\dfrac{\tan\gamma \,.\, \cot\delta - 1}{\tan\gamma + \cot\delta} = \tan(\gamma - \delta).$

(19) $\dfrac{\tan\gamma \,.\, \cot\delta + 1}{\cot\delta - \tan\gamma} = \tan(\gamma + \delta).$

(20) $\dfrac{\cot\delta - \cot\gamma}{\cot\gamma \,.\, \cot\delta + 1} = \tan(\gamma - \delta).$

(21) $\tan^2\alpha - \tan^2\beta = \dfrac{\sin(\alpha+\beta) \,.\, \sin(\alpha-\beta)}{\cos^2\alpha \,.\, \cos^2\beta}.$

(22) $\cot^2\alpha - \tan^2\beta = \dfrac{\cos(\alpha+\beta) \,.\, \cos(\alpha-\beta)}{\sin^2\alpha \,.\, \cos^2\beta}.$

(23) $\dfrac{\tan^2\alpha - \tan^2\beta}{1 - \tan^2\alpha \,.\, \tan^2\beta} = \tan(\alpha+\beta) \,.\, \tan(\alpha-\beta).$

(24) $\sin(\alpha+\beta) \,.\, \sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta = \cos^2\beta - \cos^2\alpha.$

(25) $\cos(\alpha+\beta) \,.\, \cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta = \cos^2\beta - \sin^2\alpha.$

(26) $\sin(A - 45^0) = \dfrac{\sin A - \cos A}{\sqrt{2}}.$

(27) $\sqrt{2} \,.\, \sin(A + 45^0) = \sin A + \cos A.$

(28) $\cos A - \sin A = \sqrt{2} \,.\, \cos(A + 45^0).$

(29) $\cos(A + 45^0) + \sin(A - 45^0) = 0.$

(30) $\cos(A - 45^0) = \sin(A + 45^0).$

(31) $\sin(\theta+\phi) \,.\, \cos\theta - \cos(\theta+\phi) \,.\, \sin\theta = \sin\phi.$

(32) $\sin(\theta-\phi) \,.\, \cos\phi + \cos(\theta-\phi) \,.\, \sin\phi = \sin\theta.$

(33) $\cos(\theta+\phi) \,.\, \cos\theta + \sin(\theta+\phi) \,.\, \sin\theta = \cos\phi.$

(34) $\dfrac{\tan(\theta-\phi) + \tan\phi}{1 - \tan(\theta-\phi) \,.\, \tan\phi} = \tan\theta.$

(35) $\dfrac{\tan(\theta+\phi) - \tan\theta}{1 + \tan(\theta+\phi) \,.\, \tan\theta} = \tan\phi.$

(36) $2\sin\left(\alpha + \dfrac{\pi}{4}\right) . \cos\left(\beta - \dfrac{\pi}{4}\right) = \cos(\alpha-\beta) + \sin(\alpha+\beta).$

(37) $2\sin\left(\dfrac{\pi}{4} - \alpha\right) . \cos\left(\dfrac{\pi}{4} + \beta\right) = \cos(\alpha-\beta) - \sin(\alpha+\beta).$

(38) $\cos(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\left(\dfrac{\pi}{4} + \alpha\right) . \cos\left(\dfrac{\pi}{4} + \beta\right).$

(39) $\cos(\alpha+\beta) - \sin(\alpha-\beta) = 2\sin\left(\dfrac{\pi}{4} - \alpha\right) . \cos\left(\dfrac{\pi}{4} - \beta\right).$

(40) $\sin nA \,.\, \cos A + \cos nA \,.\, \sin A = \sin (n+1) A$.

(41) $\cos (n-1) A \,.\, \cos A - \sin (n-1) A \,.\, \sin A = \cos nA$.

(42) $\sin nA \,.\, \cos (n-1) A - \cos nA \,.\, \sin (n-1) A = \sin A$.

(43) $\cos (n-1) A \,.\, \cos (n+1) A - \sin (n-1) A \,.\, \sin (n+1) A = \cos 2nA$.

156. The following formulæ are important:

$$\left.\begin{aligned} \tan (A+B) &= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B} \\ \tan (A-B) &= \frac{\tan A - \tan B}{1 + \tan A \,.\, \tan B} \end{aligned}\right\} \text{.........} \text{(ii)}.$$

$$\tan (A + 45^\circ) = \frac{\tan A + 1}{1 - \tan A},$$

$$\tan (A - 45^\circ) = \frac{\tan A - 1}{1 + \tan A}.$$

The proof of the first is given below. The student should prove the second in a similar manner. The third and fourth follow at once from the first two by putting 45° for B.

Example. To prove $\tan (A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$.

(i) By using the results of Art. 154, we have

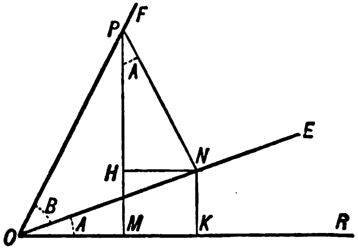
$$\tan (A+B) = \frac{\sin (A+B)}{\cos (A+B)} = \frac{\sin A \,.\, \cos B + \cos A \,.\, \sin B}{\cos A \,.\, \cos B - \sin A \,.\, \sin B}.$$

Divide the numerator and the denominator of this fraction each by $\cos A \,.\, \cos B$, and we get

$$\tan (A+B) = \frac{\dfrac{\sin A \,.\, \cos B}{\cos A \,.\, \cos B} + \dfrac{\cos A \,.\, \sin B}{\cos A \,.\, \cos B}}{\dfrac{\cos A \,.\, \cos B}{\cos A \,.\, \cos B} - \dfrac{\sin A \,.\, \sin B}{\cos A \,.\, \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}. \quad \text{Q.E.D.}$$

(ii)* * By Geometry. Make the construction of page 118;



Then $\tan(A+B)=\tan ROF=\dfrac{MP}{OM}=\dfrac{MH+HP}{OK-MK}$

$$=\frac{\dfrac{KN}{OK}+\dfrac{HP}{OK}}{\dfrac{OK}{OK}-\dfrac{HN}{OK}}=\frac{\dfrac{KN}{OK}+\dfrac{HP}{OK}}{1-\dfrac{HN\,.\,\mathrm{HP}}{\mathrm{HP}\,.\,OK}}=\frac{\tan A+\dfrac{HP}{OK}}{1-\tan A\,.\,\dfrac{HP}{OK}}.$$

Now the triangles NOK and NPH are similar,

$$\therefore \frac{HP}{OK}=\frac{NP}{ON}=\tan B,$$

$$\therefore \tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\,.\,\tan B}. \qquad \text{Q.E.D.}$$

EXAMPLES. XXXVI.

(1) If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{4}$, prove that $\tan(A+B)=\frac{6}{7}$, and $\tan(A-B)=\frac{2}{9}$.

(2) If $\tan A=1$ and $\tan B=\dfrac{1}{\sqrt{3}}$, prove that

$$\tan(A+B)=2+\sqrt{3}.$$

(3) Prove that $\tan 15^0=2-\sqrt{3}$.

(4) If $\tan A=\frac{5}{6}$ and $\tan B=\frac{1}{11}$, prove that $\tan(A+B)=1$. What is $(A+B)$ in this case?

(5) If $\tan A = m$ and $\tan B = \frac{1}{m}$, prove that $\tan(A+B) = \infty$. What is $(A+B)$ in this case?

Prove the following statements:

(6) $\cot(A+B) = \dfrac{\cot A \,.\, \cot B - 1}{\cot A + \cot B}$.

(7) $\cot(A-B) = \dfrac{\cot A \,.\, \cot B + 1}{\cot B - \cot A}$.

(8) $\cot\left(\theta - \frac{\pi}{4}\right) = \dfrac{\cot\theta + 1}{1 - \cot\theta}$.

(9) $\dfrac{\cot\theta - 1}{\cot\theta + 1} = \cot\left(\theta + \frac{\pi}{4}\right)$.

(10) $\tan\left(\theta - \frac{\pi}{4}\right) + \cot\left(\theta + \frac{\pi}{4}\right) = 0$.

(11) $\cot\left(\theta - \frac{\pi}{4}\right) + \tan\left(\theta + \frac{\pi}{4}\right) = 0$.

(12) If $\tan\alpha = \dfrac{m}{m+1}$ and $\tan\beta = \dfrac{1}{2m+1}$, prove that

$$\tan(\alpha+\beta) = 1.$$

(13) $\dfrac{\tan(n+1)\phi - \tan n\phi}{1 + \tan(n+1)\phi \,.\, \tan n\phi} = \tan\phi$.

(14) $\dfrac{\tan(n+1)\phi + \tan(1-n)\phi}{1 - \tan(n+1)\phi \,.\, \tan(1-n)\phi} = \tan 2\phi$.

(15) If $\tan\alpha = m$ and $\tan\beta = n$, prove that

$$\cos(\alpha+\beta) = \frac{1 - mn}{\sqrt{(1+m^2)(1+n^2)}}.$$

(16) If $\tan\alpha = (a+1)$ and $\tan\beta = (a-1)$, then $2\cot(\alpha-\beta) = a^2$.

(17) If $\alpha + \beta + \gamma = 90^0$, then

$$\tan\gamma = \frac{1 - \tan\alpha\tan\beta}{\tan\alpha + \tan\beta}.$$

157. From pages 118 and 119 we have

$$\left.\begin{aligned}\sin(A+B)&=\sin A\,.\,\cos B+\cos A\,.\,\sin B\\ \sin(A-B)&=\sin A\,.\,\cos B-\cos A\,.\,\sin B\\ \cos(A+B)&=\cos A\,.\,\cos B-\sin A\,.\,\sin B\\ \cos(A-B)&=\cos A\,.\,\cos B+\sin A\,.\,\sin B\end{aligned}\right\}\ldots\text{(i).}$$

From these by addition and subtraction we get

$$\left.\begin{aligned}\sin(A+B)+\sin(A-B)&=2\sin A\,.\,\cos B\\ \sin(A+B)-\sin(A-B)&=2\cos A\,.\,\sin B\\ \cos(A+B)+\cos(A-B)&=2\cos A\,.\,\cos B\\ \cos(A-B)-\cos(A+B)&=2\sin A\,.\,\sin B\end{aligned}\right\}$$

Now put S for $(A+B)$,

and put T for $(A-B)$

Then $S+T=2A$, and $S-T=2B$,

so that $A=\dfrac{S+T}{2}$, and $B=\dfrac{S-T}{2}$.

Hence the above results may be written

$$\left.\begin{aligned}\sin S+\sin T&=2\sin\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}\\ \sin S-\sin T&=2\cos\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}\\ \cos S+\cos T&=2\cos\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}\\ {}^{*}\cos T-\cos S&=2\sin\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}\end{aligned}\right\}\ldots\text{(iii).}$$

---

* If A and B are each less than 90°, then S, which is their *sum*, is greater than T, their *difference*. Therefore if S be less than 90°, $\cos S$ is less than $\cos T$; so that $\cos T-\cos S$ is *positive*.

158. The formulæ (iii) are most important, and the student is recommended to get thoroughly familiar with them *in words*, thus:

(1) *The sum of the sines of two angles is equal to twice the sine of half their sum multiplied by the cosine of half their difference.*

(2) *The sine of the greater of two angles minus the sine of the lesser is equal to twice the cosine of half their sum multiplied by the sine of half their difference.*

(3) *The sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.*

(4) *The cosine of the lesser of two angles minus the cosine of the greater is equal to twice the sine of half their sum multiplied by the sine of half their difference.*

159. It will be convenient to refer to the formulæ (i) as the 'A,B' formulæ, and to the formulæ (iii) as the 'S,T' formulæ.

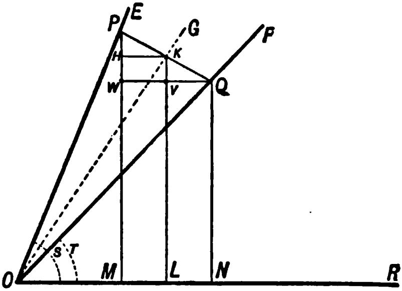
160. The 'S,T' formulæ can be proved directly from a figure.

We give the proof for the case in which S and T are each less than 90°.

On first reading the subject however the student should omit the next two pages, and all other alternative proofs.

**(1) *To prove that*

$$\sin S+\sin T=2\sin\frac{S+T}{2}\,.\cos\frac{S-T}{2}.$$



In the figure, let ROE be the angle S, ROF the angle T; then FOE is the angle $S-T$. Let OG bisect the angle EOF.

Then the angle $FOG=\frac{FOE}{2}=\frac{S-T}{2}$.

Also the angle $ROG=ROF+FOG=T+\frac{S-T}{2}=\frac{S+T}{2}$.

In OE take any point P, and from OF cut off OQ equal to OP. Join PQ, cutting OG in K. Then OG which bisects the vertical angle of the isosceles triangle POQ, bisects PQ at right angles. Draw PM, KL, QN at right angles to OR, and draw QVW and KH parallel to NLM.

Then by parallels, since $PK=KQ$, therefore $PH=HW$ and $WV=VQ$. Hence,

$$\sin S+\sin T=\frac{MP}{OP}+\frac{NQ}{OQ}=\frac{MP+NQ}{OQ}$$

$$=\frac{(LK+HP)+(LK-WH)}{OQ}=\frac{2LK}{OQ}$$

$$=\frac{2LK\,.\,\mathrm{OK}}{\mathrm{OK}\,.\,OQ}=2\sin ROK\,.\cos QOK$$

$$=2\sin\frac{S+T}{2}\,.\cos\frac{S-T}{2}.\quad\text{Q. E. D.}$$

(2) *To prove*

$$\cos S+\cos T=2\cos\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}.$$

With the same figure and construction, we have

$$\begin{aligned}\cos S+\cos T&=\frac{OM}{OP}+\frac{ON}{OQ}=\frac{OM+ON}{OQ},\\ &=\frac{(OL-ML)+(OL+LN)}{OQ}=\frac{2OL}{OQ},\\ &=\frac{2OL\,.\,\mathrm{OK}}{\mathrm{OK}\,.\,OQ}=2\cos ROK\,.\,\cos QOK,\\ &=2\cos\frac{S+T}{2}\,.\,\cos\frac{S-T}{2}.\quad\text{Q.E.D.}\end{aligned}$$

(3) *To prove*

$$\sin S-\sin T=2\sin\frac{S-T}{2}\,.\,\cos\frac{S+T}{2}.$$

In the above figure the angles VKQ, ROK are each the complement of LKO $\quad\therefore\ VKQ=ROK=\frac{S+T}{2}$. Hence

$$\begin{aligned}\sin S-\sin T&=\frac{MP}{OP}-\frac{NQ}{OQ}=\frac{MP-NQ}{OQ},\\ &=\frac{(LK+HP)-(LK-WH)}{OQ}=\frac{2VK}{OQ},\\ &=\frac{2VK\,.\,\mathrm{KQ}}{\mathrm{KQ}\,.\,OQ}=2\cos VKQ\,.\,\sin QOK,\\ &=2\sin\frac{S-T}{2}\,.\,\cos\frac{S+T}{2}.\quad\text{Q.E.D.}\end{aligned}$$

(4) *To prove*

$$\cos T-\cos S=2\sin\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}.$$

With the same figure and construction, we have

$$\begin{aligned}\cos T-\cos S&=\frac{ON}{OQ}-\frac{OM}{OP}=\frac{ON-OM}{OQ},\\ &=\frac{(OL+LN)-(OL-ML)}{OQ}=\frac{2LN}{OQ},\\ &=\frac{2VQ\,.\,\mathrm{KQ}}{\mathrm{KQ}\,.\,OQ}=2\sin VKQ\,.\,\sin QOK,\\ &=2\sin\frac{S+T}{2}\,.\,\sin\frac{S-T}{2}.\quad\text{Q.E.D.}\end{aligned}$$

EXAMPLES. XXXVII.

Prove the following statements:

(1) $\sin 60^0 + \sin 30^0 = 2 \sin 45^0 . \cos 15^0$.

(2) $\sin 60^0 + \sin 20^0 = 2 \sin 40^0 . \cos 20^0$.

(3) $\sin 40^0 - \sin 10^0 = 2 \cos 25^0 . \sin 15^0$.

(4) $\cos \frac{\pi}{3} + \cos \frac{\pi}{2} = 2 \cos \frac{5\pi}{12} . \cos \frac{\pi}{12}$.

(5) $\cos \frac{\pi}{3} - \cos \frac{\pi}{2} = 2 \sin \frac{5\pi}{12} . \sin \frac{\pi}{12}$.

(6) $\sin 3A + \sin 5A = 2 \sin 4A . \cos A$.

(7) $\sin 7A - \sin 5A = 2 \cos 6A . \sin A$.

(8) $\cos 5A + \cos 9A = 2 \cos 7A . \cos 2A$.

(9) $\cos 5A - \cos 4A = -2 \sin \frac{9A}{2} . \sin \frac{A}{2}$.

(10) $\cos A - \cos 2A = 2 \sin \frac{3A}{2} . \sin \frac{A}{2}$.

(11) $\frac{\sin 2\theta + \sin \theta}{\cos \theta + \cos 2\theta} = \tan \frac{3\theta}{2}$.

(12) $\frac{\sin 2\theta - \sin \theta}{\cos \theta - \cos 2\theta} = \cot \frac{3\theta}{2}$.

(13) $\frac{\sin 3\theta + \sin 2\theta}{\cos 2\theta - \cos 3\theta} = \cot \frac{\theta}{2}$.

(14) $\frac{\sin \theta + \sin \phi}{\cos \theta - \cos \phi} = \frac{\cos \theta + \cos \phi}{\sin \phi - \sin \theta}$.

(15) $\cos (60^0 + A) + \cos (60^0 - A) = \cos A$.

(16) $\cos (45^0 + A) + \cos (45^0 - A) = \sqrt{2} . \cos A$.

(17) $\sin (45^0 + A) - \sin (45^0 - A) = \sqrt{2} . \sin A$.

(18) $\cos (30^0 - A) - \cos (30^0 + A) = \sin A$.

(19) $\frac{\sin \theta - \sin \phi}{\cos \phi - \cos \theta} = \cot \frac{\theta + \phi}{2}$.

(20) $\frac{\sin \theta - \sin \phi}{\sin \theta + \sin \phi} = \cot \left(\frac{\theta + \phi}{2}\right) . \tan \left(\frac{\theta - \phi}{2}\right)$.

161. It is important that the student should be thoroughly familiar with the second set of formulæ on p. 126.

Written as follows, they may be regarded as the inverse of the '*S*, *T*' formulæ.

$$\left.\begin{aligned} 2\sin A \,.\, \cos B &= \sin(A+B)+\sin(A-B), \\ 2\cos A \,.\, \sin B &= \sin(A+B)-\sin(A-B), \\ 2\cos A \,.\, \cos B &= \cos(A+B)+\cos(A-B), \\ 2\sin A \,.\, \sin B &= \cos(A-B)-\cos(A+B). \end{aligned}\right\} \text{iv.}$$

EXAMPLES. XXXVIII.

Express as the sum or as the difference of two trigonometrical ratios the ten following expressions:

(1) $2\sin\theta \,.\, \cos\phi$.
(2) $2\cos\alpha \,.\, \cos\beta$.
(3) $2\sin 2\alpha \,.\, \cos 3\beta$.
(4) $2\cos(\alpha+\beta) \,.\, \cos(\alpha-\beta)$.
(5) $2\sin 3\theta \,.\, \cos 5\theta$.
(6) $2\cos\frac{3\theta}{2} \,.\, \cos\frac{\theta}{2}$.
(7) $\sin 4\theta \,.\, \sin\theta$.
(8) $\cos\frac{5\theta}{2} \,.\, \sin\frac{3\theta}{2}$.
(9) $2\cos 10^\circ \,.\, \sin 50^\circ$.
(10) $\cos 45^\circ \,.\, \sin 15^\circ$.

(11) Simplify $2\cos 2\theta \,.\, \cos\theta - 2\sin 4\theta \,.\, \sin\theta$.

(12) Simplify $\sin\frac{5\theta}{2} \,.\, \cos\frac{\theta}{2} - \sin\frac{9\theta}{2} \,.\, \cos\frac{3\theta}{2}$

(13) Simplify $\sin 3\theta + \sin 2\theta + 2\sin\frac{3\theta}{2} \,.\, \cos\frac{\theta}{2}$.

(14) Prove that $\sin\frac{11\theta}{4} \,.\, \sin\frac{\theta}{4} + \sin\frac{7\theta}{4} \,.\, \sin\frac{3\theta}{4} = \sin 2\theta \,.\, \sin\theta$.

* * MISCELLANEOUS EXAMPLES. XXXIX.

(1) If $\tan\alpha=\frac{1}{2}$ and $\tan\beta=\frac{1}{3}$, prove that $\tan(\alpha+\beta)=1$.

(2) If $\tan\alpha=\frac{3}{4}$ and $\tan\beta=\frac{1}{7}$, prove that one of the values of $\alpha+\beta$ is $\frac{\pi}{4}$.

(3) If $\tan\alpha=\dfrac{m-1}{m}$ and $\tan\beta=\dfrac{1}{2m-1}$, shew that one value of $(\alpha+\beta)$ is $\frac{\pi}{4}$.

(4) Simplify $\dfrac{\cos\alpha-\cos 5\alpha}{\sin\alpha+\sin 5\alpha}$.

(5) Simplify $\dfrac{\sin 5x-\sin 3x}{\cos 5x+\cos 3x}$.

(6) Prove that $\dfrac{\cos A+\cos 3A}{\cos 3A+\cos 5A}=\dfrac{\cos 2A}{\cos 4A}$.

(7) Simplify $\dfrac{\sin 3x-\sin x}{\cos 3x+\cos x}+\dfrac{\sin 3x+\sin x}{\cos 3x-\cos x}$.

(8) Simplify $\dfrac{(\sin 4A-\sin 2A)(\cos A-\cos 3A)}{(\cos 4A+\cos 2A)(\sin A+\sin 3A)}$.

(9) Prove that

$$2\sin 2\alpha\,.\,\cos\alpha+2\cos 4\alpha\,.\,\sin\alpha=\sin 5\alpha+\sin\alpha.$$

(10) Prove that

$$\cos 2\alpha\,.\,\cos\alpha-\sin 4\alpha\,.\,\sin\alpha=\cos 3\alpha\,.\,\cos 2\alpha.$$

(11) $\tan 2A\,.\,\tan 3A\,.\,\tan 5A=\tan 5A-\tan 3A-\tan 2A$.

(12) Solve $\left.\begin{array}{l}4\sin(\theta+\phi)\,.\,\cos(\theta-\phi)=3\\ 4\cos(\theta+\phi)\,.\,\sin(\theta-\phi)=1\end{array}\right\}$.

(13) Prove that

$$\frac{\sin A\,.\,\sin 2A+\sin 2A\,.\,\sin 5A+\sin 3A\,.\,\sin 10A}{\cos A\,.\,\sin 2A+\sin 2A\,.\,\cos 5A-\cos 3A\,.\,\sin 10A}=-\tan 7A.$$

(14) $\tan\dfrac{A+B}{2}-\tan\dfrac{A-B}{2}=\dfrac{2\sin B}{\cos A+\cos B}$.

CHAPTER XII.

ON THE TRIGONOMETRICAL RATIOS OF MULTIPLE ANGLES.

162. To express the Trigonometrical Ratios of the angle $2A$ in terms of those of the angle A.

Since $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$;

$$\therefore \sin(A+A) = \sin A \cdot \cos A + \cos A \cdot \sin A;$$

$$\therefore \sin 2A = 2 \sin A \cdot \cos A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Also, since $\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B$;

$$\therefore \cos(A+A) = \cos A \cdot \cos A - \sin A \cdot \sin A;$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A \quad \ldots\ldots\ldots\ldots \quad (2).$$

But $$1 = \cos^2 A + \sin^2 A;$$

$$\therefore 1 + \cos 2A = 2\cos^2 A,$$

$$\text{and } 1 - \cos 2A = 2\sin^2 A.$$

The last two results are usually written

$$\cos 2A = 2\cos^2 A - 1 \quad \ldots\ldots\ldots\ldots\ldots \quad (3),$$

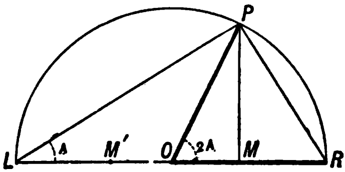
$$\text{and } \cos 2A = 1 - 2\sin^2 A \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

Again, $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$;

$$\therefore \tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \cdot \tan A};$$

$$\therefore \tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

163.* * *To prove the ‘2A’ formulæ geometrically.*



Let ROP be the angle $2A$. With centre O and any radius describe the semicircle RPL. Draw PM perpendicular to OR. Join RP . PL.

Then the angle RPL in a semicircle is a right angle. The angle $ROP = OLP + OPL = 2\,OLP$ [since $OL = OP$]. $\therefore$ $OLP =$ a half of $ROP = A$. Also MPR and OLP are each the complement of MPL. $\therefore$ $MPR = OLP = A$.

Hence

(1) $$\sin 2A = \frac{MP}{OP} = \frac{2MP}{2OR} = \frac{2MP}{LR} = \frac{2MP \,.\, \mathrm{PR}}{\mathrm{PR} \,.\, LR}$$
$$= 2\cos MPR \,.\, \sin PLR = 2\sin A \,.\, \cos A.$$

(3) $$\cos 2A = \frac{OM}{OP} = \frac{LM - LO}{OP} = \frac{2LM}{2OP} - \frac{LO}{OP}$$
$$= \frac{2LM \,.\, \mathrm{LP}}{\mathrm{LP} \,.\, LR} - \frac{OP}{OP} = 2\cos MLP \,.\, \cos PLR - 1$$
$$= 2\cos^2 A - 1.$$

(2) Let $OM' = OM$.

Then $$2OM = M'M = LM - LM' = LM - MR.$$ Hence,
$$\cos 2A = \frac{2OM}{2OP} = \frac{LM - MR}{LR} = \frac{LM}{LR} - \frac{MR}{LR},$$
$$= \frac{LM \,.\, \mathrm{LP}}{\mathrm{LP} \,.\, LR} - \frac{MR \,.\, \mathrm{PR}}{\mathrm{PR} \,.\, LR} = \cos^2 A - \sin^2 A.$$

(5) $$\tan 2A = \frac{2MP}{2OM} = \frac{2MP}{LM - MR},$$
$$= \frac{\dfrac{2MP}{LM}}{\dfrac{LM}{LM} - \dfrac{MR}{LM}} = \frac{2\tan A}{1 - \dfrac{MR \,.\, MP}{MP \,.\, LM}} = \frac{2\tan A}{1 - \tan^2 A}.$$

164. These five formulæ are very important,

$$\left.\begin{array}{ll} \sin 2A = 2\sin A \,.\, \cos A & \ldots\ldots\ldots\ldots\ldots\ldots (1), \\ \left.\begin{array}{l} \cos 2A = \cos^2 A - \sin^2 A \\ \cos 2A = 2\cos^2 A - 1 \\ \cos 2A = 1 - 2\sin^2 A \end{array}\right\} & \begin{array}{r} \ldots\ldots\ldots\ldots (2), \\ \ldots\ldots\ldots\ldots (3), \\ \ldots\ldots\ldots\ldots (4), \end{array} \\ \tan 2A = \dfrac{2\tan A}{1-\tan^2 A} & \ldots\ldots\ldots\ldots\ldots\ldots (5), \end{array}\right\} \text{v.}$$

165. The following result is important,

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A \,.\, \cos A}{2\cos^2 A} = \tan A.$$

166. The student must notice that A is *any* angle, and therefore these formulæ will be true whatever we put for A.

Example. Write $\frac{A}{2}$ instead of A, and we get

$$\sin A = 2\sin\frac{A}{2} \,.\, \cos\frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and so on.

EXAMPLES. XL.

Prove the following statements:

(1) $2\operatorname{cosec} 2A = \sec A \,.\, \operatorname{cosec} A.$

(2) $\dfrac{\operatorname{cosec}^2 A}{\operatorname{cosec}^2 A - 2} = \sec 2A.$

(3) $\dfrac{2-\sec^2 A}{\sec^2 A} = \cos 2A.$

(4) $\cos^2 A\,(1-\tan^2 A) = \cos 2A.$

(5) $\cot 2A = \dfrac{\cot^2 A - 1}{2\cot A}.$

(6) $\dfrac{2\tan B}{1+\tan^2 B}=\sin 2B.$

(7) $\tan B+\cot B=2\operatorname{cosec} 2B.$

(8) $\dfrac{1-\tan^2 B}{1+\tan^2 B}=\cos 2B.$

(9) $\cot B-\tan B=2\cot 2B.$

(10) $\dfrac{\cot^2 B+1}{\cot^2 B-1}=\sec 2B.$

(11) $\left(\sin\dfrac{\theta}{2}+\cos\dfrac{\theta}{2}\right)^2=1+\sin\theta.$

(12) $\left(\sin\dfrac{\theta}{2}-\cos\dfrac{\theta}{2}\right)^2=1-\sin\theta.$

(13) $\cos^2\dfrac{\theta}{2}\left(1+\tan\dfrac{\theta}{2}\right)^2=1+\sin\theta.$

(14) $\sin^2\dfrac{\theta}{2}\left(\cot\dfrac{\theta}{2}-1\right)^2=1-\sin\theta.$

(15) $\left(\dfrac{\tan\dfrac{\theta}{2}+1}{\tan\dfrac{\theta}{2}-1}\right)^2=\dfrac{1+\sin\theta}{1-\sin\theta}.$

(16) $\dfrac{\sin\beta}{1+\cos\beta}=\tan\dfrac{\beta}{2}.$

(17) $\dfrac{\sin\beta}{1-\cos\beta}=\cot\dfrac{\beta}{2}.$

(18) $\dfrac{1-\cos\beta}{1+\cos\beta}=\tan^2\dfrac{\beta}{2}.$

(19) $\dfrac{1+\sec\beta}{\sec\beta}=2\cos^2\dfrac{\beta}{2}.$

(20) $\operatorname{cosec}\beta-\cot\beta=\tan\dfrac{\beta}{2}.$

(21) $\dfrac{\cos 2x}{1+\sin 2x}=\dfrac{1-\tan x}{1+\tan x}.$

(22) $\dfrac{\cos x}{1-\sin x}=\dfrac{1+\tan\dfrac{x}{2}}{1-\tan\dfrac{x}{2}}.$

(23) $\dfrac{\cos x}{1+\sin x}=\dfrac{\cot\frac{x}{2}-1}{\cot\frac{x}{2}+1}.$

(24) $\dfrac{\cos x}{1-\sin x}=\dfrac{\cot\frac{x}{2}+1}{\cot\frac{x}{2}-1}.$

(25) $\dfrac{1+\sin x+\cos x}{1+\sin x-\cos x}=\cot\dfrac{x}{2}.$

(26) $\dfrac{\cos^3 a+\sin^3 a}{\cos a+\sin a}=\dfrac{2-\sin 2a}{2}.$

(27) $\dfrac{\cos^3 a-\sin^3 a}{\cos a-\sin a}=\dfrac{2+\sin 2a}{2}.$

(28) $\cos^4 a-\sin^4 a=\cos 2a.$

(29) $\cos^6 a+\sin^6 a=\dfrac{1+3\cos^2 2a}{4}.$

(30) $\cos^6 a-\sin^6 a=\dfrac{(3+\cos^2 2a)\cos 2a}{4}.$

(31) $\dfrac{\sin 3\beta}{\sin\beta}-\dfrac{\cos 3\beta}{\cos\beta}=2.$

(32) $\dfrac{\cos 3\beta}{\sin\beta}+\dfrac{\sin 3\beta}{\cos\beta}=2\cot 2\beta.$

(33) $\dfrac{\sin 4\beta}{\sin 2\beta}=2\cos 2\beta.$

(34) $\dfrac{\sin 5\beta}{\sin\beta}-\dfrac{\cos 5\beta}{\cos\beta}=4\cos 2\beta.$

(35) $\dfrac{\sin\frac{5\pi}{12}}{\sin\frac{\pi}{12}}-\dfrac{\cos\frac{5\pi}{12}}{\cos\frac{\pi}{12}}=2\sqrt{3}.$

(36) $\tan(45^\circ+A)-\tan(45^\circ-A)=2\tan 2A.$

(37) $\tan(45^\circ-A)+\cot(45^\circ-A)=2\sec 2A.$

(38) $\dfrac{\tan^2(45^0+A)-1}{\tan^2(45^0+A)+1}=\sin 2A.$

(39) $\dfrac{\sec A+\tan A}{\sec A-\tan A}=\tan\left(45^0+\dfrac{A}{2}\right).\cot\left(45^0-\dfrac{A}{2}\right).$

(40) $\dfrac{\cos(A+45^0)}{\cos(A-45^0)}=\sec 2A-\tan 2A.$

(41) $\tan B=\dfrac{\sin B+\sin 2B}{1+\cos B+\cos 2B}.$

(42) $\tan B=\dfrac{\sin 2B-\sin B}{1-\cos B+\cos 2B}.$

167. The following two formulæ should be remembered:

$$\left.\begin{array}{l}\sin 3A=3\sin A-4\sin^3 A\\ \cos 3A=4\cos^3 A-3\cos A\end{array}\right\}\ldots\ldots\ldots\ldots\ldots(\text{vi}).$$

NOTE. The similarity of these two results is apt to cause confusion. This may be avoided by observing that the second formula must be true when $A=0^0$; and then $\cos 3A=\cos 0^0=1$. In which case the formula gives $\cos 0^0=4\cos 0^0-3\cos 0^0$, or $1=4-3$, which is true.

The first formula may be proved thus:

$$\begin{aligned}\sin 3A=\sin(2A+A)&=\sin 2A.\cos A+\cos 2A.\sin A\\ &=(2\sin A.\cos A)\cos A+(1-2\sin^2 A)\sin A\\ &=2\sin A.\cos^2 A+\sin A-2\sin^3 A\\ &=2\sin A(1-\sin^2 A)+\sin A-2\sin^3 A\\ &=2\sin A-2\sin^3 A+\sin A-2\sin^3 A\\ &=3\sin A-4\sin^3 A.\end{aligned}$$

The second formula may be proved in a similar manner.

Example. Prove that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

$$\tan 3A = \frac{\sin 3A}{\cos 3A} = \frac{3\sin A - 4\sin^3 A}{4\cos^3 A - 3\cos A}$$

$$= \frac{3\sin A\,(\sin^2 A + \cos^2 A) - 4\sin^3 A}{4\cos^3 A - 3\cos A\,(\sin^2 A + \cos^2 A)}$$

$$= \frac{3\sin A\,.\cos^2 A - \sin^3 A}{\cos^3 A - 3\cos A\,.\sin^2 A}$$

$$= \frac{\dfrac{3\sin A\,.\cos^2 A}{\cos^3 A} - \dfrac{\sin^3 A}{\cos^3 A}}{\dfrac{\cos^3 A}{\cos^3 A} - \dfrac{3\cos A\,.\sin^2 A}{\cos^3 A}} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

EXAMPLES. XLI.

Prove the following statements:

(1) $\dfrac{\sin 3A}{\sin A} = 2\cos 2A + 1.$

(2) $\dfrac{\cos 3A}{\cos A} = 2\cos 2A - 1.$

(3) $\dfrac{3\sin A - \sin 3A}{\cos 3A + 3\cos A} = \tan^3 A.$

(4) $\cot 3A = \dfrac{\cot^3 A - 3\cot A}{3\cot^2 A - 1}.$

(5) $\dfrac{\sin 3A - \sin A}{\cos 3A + \cos A} = \tan A.$

(6) $\dfrac{\sin 3A - \cos 3A}{\sin A + \cos A} = 2\sin 2A - 1.$

(7) $\dfrac{\sin 3A + \cos 3A}{\cos A - \sin A} = 2\sin 2A + 1.$

(8) $\dfrac{1}{\tan 3A - \tan A} + \dfrac{1}{\cot A - \cot 3A} = \cot 2A.$

(9) $\left(\dfrac{3\sin A - \sin 3A}{3\cos A + \cos 3A}\right)^2 = \left(\dfrac{\sec 2A - 1}{\sec 2A + 1}\right)^3.$

(10) $\dfrac{1 - \cos 3A}{1 - \cos A} = (1 + 2\cos A)^2.$

**MISCELLANEOUS EXAMPLES. XLII.

Prove the following statements :

(1) $\dfrac{\sin A+\cos A}{\cos A-\sin A}=\tan 2A+\sec 2A.$

(2) $\dfrac{\tan\dfrac{A}{2}+1}{1-\tan\dfrac{A}{2}}=\tan A+\sec A.$

(3) $\sin(n+1)\alpha.\cos(n-1)\alpha-\sin 2\alpha=\sin(n-1)\alpha.\cos(n+1)\alpha.$

(4) $\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}=\tan\dfrac{\alpha+\beta}{2}.$

(5) $\dfrac{\cos 2\alpha+\cos 12\alpha}{\cos 6\alpha+\cos 8\alpha}+\dfrac{\cos 7\alpha-\cos 3\alpha}{\cos\alpha-\cos 3\alpha}+2\dfrac{\sin 4\alpha}{\sin 2\alpha}=0.$

(6) If $A=18^0$, prove that $\sin 2A=\cos 3A$; hence prove that

$$\sin 18^0=\frac{\sqrt{5}-1}{4}.$$

(7) $\dfrac{\sin\alpha+\sin\beta+\sin(\alpha+\beta)}{\sin\alpha+\sin\beta-\sin(\alpha+\beta)}=\cot\dfrac{\alpha}{2}.\cot\dfrac{\beta}{2}.$

(8) $\sin 2A.\sin 2B=\sin^2(A+B)-\sin^2(A-B).$

(9) $\cos 4A=8\cos^4 A-8\cos^2 A+1.$

(10) $\tan 50^0+\cot 50^0=2\sec 10^0.$

(11) $\sin 3A=4\sin A.\sin(60^0+A)\sin(60^0-A).$

(12) $\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)^2(\cot A-2\cot 2A)=4\cot A.$

(13) $\dfrac{\cos 3\alpha-\sin\beta.\sin 5\alpha-\cos 7\alpha}{\sin 3\alpha+\sin\beta.\cos 5\alpha-\sin 7\alpha}$ is independent of β.

(14) $(\cos x+\cos y)^2+(\sin x+\sin y)^2=4\cos^2\dfrac{x-y}{2}.$

(15) $2\cos^2 A.\cos^2 B+2\sin^2 A.\sin^2 B=1+\cos 2A.\cos 2B.$

(16) $\cot\dfrac{\pi}{8}-\tan\dfrac{\pi}{8}=2.$

(17) $\tan 4\theta=\dfrac{4\tan\theta(1-\tan^2\theta)}{1-6\tan^2\theta+\tan^4\theta}.$

(18) $2\cos\frac{\pi}{8}=\sqrt{2+\sqrt{2}}.$

(19) $2\cos 11^0\,15'=\sqrt{2+\sqrt{2+\sqrt{2}}}.$

(20) $\dfrac{\sin A\,.\,\sin 2A+\sin A\,.\,\sin 4A+\sin 2A\,.\,\sin 7A}{\sin A\,.\,\cos 2A+\sin 2A\,.\,\cos 5A+\sin A\,.\,\cos 8A}=\tan 5A.$

(21) $\dfrac{\sin\theta+\sin(\theta+\phi)+\sin(\theta+2\phi)}{\cos\theta+\cos(\theta+\phi)+\cos(\theta+2\phi)}=\tan(\theta+\phi).$

(22) $2\cos^8 A-2\sin^8 A=\cos 2A\,(1+\cos^2 2A).$

(23) $(3\sin A-4\sin^3 A)^2+(4\cos^3 A-3\cos A)^2=1.$

(24) $\dfrac{\sin 2\alpha\,.\,\cos\alpha}{(1+\cos 2\alpha)\,(1+\cos\alpha)}=\tan\dfrac{\alpha}{2}.$

(25) $2\,\dfrac{\cot(n-2)\alpha\,.\,\cot n\alpha+1}{\cot(n-2)\alpha-\cot n\alpha}=\cot\alpha-\tan\alpha.$

(26) If $\tan\alpha=\frac{1}{7}$ and $\tan\beta=\frac{2}{11}$, prove $\tan(2\alpha+\beta)=\frac{1}{2}$.

(27) Prove that $\tan\dfrac{A}{2}$ and $\cot\dfrac{A}{2}$ are the roots of the equation

$$x^2-2x\,.\,\operatorname{cosec} A+1=0.$$

(28) If $\tan B=\dfrac{b}{a}$, prove that $\sqrt{\dfrac{a+b}{a-b}}+\sqrt{\dfrac{a-b}{a+b}}=\dfrac{2\cos B}{\sqrt{\cos 2B}}.$

*168. The following examples are symmetrical, and each involve more than two angles:

Example 1. Prove that

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\sin\beta\,.\,\cos\gamma\,.\,\cos\alpha$$
$$+\sin\gamma\,.\,\cos\alpha\,.\,\cos\beta-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.$$

$$\begin{aligned}\sin(\alpha+\beta+\gamma)&=\sin(\alpha+\beta)\,.\,\cos\gamma+\cos(\alpha+\beta)\sin\gamma\\&=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\cos\alpha\,.\,\sin\beta\,.\,\cos\gamma\\&\qquad+\cos\alpha\,.\,\cos\beta\,.\,\sin\gamma-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma\\&=\sin\alpha\,.\,\cos\beta\,.\,\cos\gamma+\sin\beta\,.\,\cos\gamma\,.\,\cos\alpha\\&\qquad+\sin\gamma\,.\,\cos\alpha\,.\,\cos\beta-\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.\end{aligned}$$

Q.E.D.

Example 2. Prove that

$$\sin a+\sin\beta+\sin\gamma-\sin(a+\beta+\gamma)$$
$$=4\,.\sin\frac{\beta+\gamma}{2}\,.\sin\frac{\gamma+a}{2}\,.\sin\frac{a+\beta}{2}.$$

Now $\sin a-\sin(a+\beta+\gamma)=-2\cos.\frac{2a+\beta+\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}.$

And $\sin\beta+\sin\gamma=2\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\beta-\gamma}{2},$ [Art. 158]

$$\therefore\ \sin a+\sin\beta+\sin\gamma-\sin(a+\beta+\gamma)$$
$$=2\sin\frac{\beta+\gamma}{2}\,.\cos\frac{\beta-\gamma}{2}-2\cos\frac{2a+\beta+\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}$$
$$=2\sin\frac{\beta+\gamma}{2}\left\{\cos\frac{\beta-\gamma}{2}-\cos\frac{2a+\beta+\gamma}{2}\right\}$$
$$=2\sin\frac{\beta+\gamma}{2}\,.\,2\sin\frac{\gamma+a}{2}\,.\sin\frac{a+\beta}{2}\qquad\text{[Art. 158]}$$
$$=4\sin\frac{\beta+\gamma}{2}\,.\sin\frac{\gamma+a}{2}\,.\sin\frac{a+\beta}{2}.\qquad\text{Q.E.D.}$$

* EXAMPLES. XLIII.

Prove the following statements :

(1) $\cos(a+\beta+\gamma)=\cos a\,.\cos\beta\,.\cos\gamma-\cos a\,.\sin\beta\,.\sin\gamma$
$-\cos\beta\,.\sin\gamma\,.\sin a-\cos\gamma\,.\sin a\,.\sin\beta.$

(2) $\sin(a+\beta-\gamma)=\sin a\,.\cos\beta\,.\cos\gamma+\sin\beta\,.\cos\gamma\,.\cos a$
$-\sin\gamma\,.\cos a\,.\cos\beta+\sin a\sin\beta\sin\gamma.$

(3) $\cos(a-\beta+\gamma)=\cos a\,.\cos\beta\,.\cos\gamma+\cos a\,.\sin\beta\,.\sin\gamma$
$-\cos\beta\,.\sin a\,.\sin\gamma+\cos\gamma\,.\sin\beta\,.\sin a.$

(4) $\sin a+\sin\beta-\sin\gamma-\sin(a+\beta-\gamma)$
$=4\sin\frac{a-\gamma}{2}\,.\sin\frac{\beta-\gamma}{2}\,.\sin\frac{a+\beta}{2}.$

(5) $\sin(a-\beta-\gamma)-\sin a+\sin\beta+\sin\gamma$
$=4\sin\frac{a-\beta}{2}\,.\sin\frac{a-\gamma}{2}\,.\sin\frac{\beta+\gamma}{2}.$

(6) $\sin 2a+\sin 2\beta+\sin 2\gamma-\sin 2(a+\beta+\gamma)$
$=4\sin(\beta+\gamma)\,.\sin(\gamma+a)\,.\sin(a+\beta).$

(7) $$\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)$$
$$+4\sin\frac{(\beta-\gamma)}{2}\,.\,\sin\frac{(\gamma-\alpha)}{2}\,.\,\sin\frac{(\alpha-\beta)}{2}=0.$$

(8) $$\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$$
$$-\sin(\alpha+\beta+\gamma)=4\sin\alpha\,.\,\sin\beta\,.\,\sin\gamma.$$

(9) $$\sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)$$
$$-\sin(\alpha+\beta-\gamma)=4\cos\alpha\,.\,\cos\beta\,.\,\sin\gamma.$$

(10) $$\cos x+\cos y+\cos z+\cos(x+y+z)$$
$$=4\cos\frac{y+z}{2}\,.\,\cos\frac{z+x}{2}\,.\,\cos\frac{x+y}{2}.$$

(11) $$\cos 2x+\cos 2y+\cos 2z+\cos 2(x+y+z)$$
$$=4\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y).$$

(12) $$\cos(y+z-x)+\cos(z+x-y)+\cos(x+y-z)$$
$$+\cos(x+y+z)=4\cos x\,.\,\cos y\,.\,\cos z.$$

(13) $$\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y+z)$$
$$=2\{1+\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$$

(14) $$\sin^2 x+\sin^2 y+\sin^2 z+\sin^2(x+y+z)$$
$$=2\{1-\cos(y+z)\,.\,\cos(z+x)\,.\,\cos(x+y)\}.$$

(15) $$\cos^2 x+\cos^2 y+\cos^2 z+\cos^2(x+y-z)$$
$$=2\{1+\cos(x-z)\,.\,\cos(y-z)\,.\,\cos(x+y)\}.$$

(16) $$\cos\alpha\,.\,\sin(\beta-\gamma)+\cos\beta\,.\,\sin(\gamma-\alpha)+\cos\gamma\,.\,\sin(\alpha-\beta)=0.$$

(17) $$\sin\alpha\,.\,\sin(\beta-\gamma)+\sin\beta\,.\,\sin(\gamma-\alpha)+\sin\gamma\,.\,\sin(\alpha-\beta)=0.$$

(18) $$\cos(\alpha+\beta)\,.\,\cos(\alpha-\beta)+\sin(\beta+\gamma)\sin(\beta-\gamma)$$
$$-\cos(\alpha+\gamma)\,.\,\cos(\alpha-\gamma)=0.$$

(19) $$\cos(\delta-\alpha)\,.\,\sin(\beta-\gamma)+\cos(\delta-\beta)\,.\,\sin(\gamma-\alpha)$$
$$-\cos(\delta-\gamma)\,.\,\sin(\beta-\alpha)=0.$$

(20) $$8\cos\frac{\theta+\phi+\chi}{2}\,.\,\cos\frac{\phi+\chi-\theta}{2}\,.\,\cos\frac{\chi+\theta-\phi}{2}\,.\,\cos\frac{\theta+\phi-\chi}{2}$$
$$=\cos 2\theta+\cos 2\phi+\cos 2\chi+4\cos\theta\,.\,\cos\phi\,.\,\cos\chi+1.$$

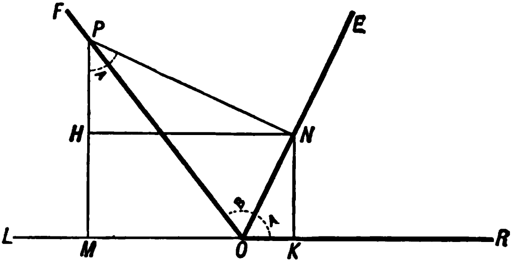
CHAPTER XIII.**

ON ANGLES UNLIMITED IN MAGNITUDE. II.

169. THE **words** of the proofs (on pages 118, 119) of the 'A, B' formulæ apply to angles of any magnitude. The **figures** will be different for angles of different magnitude.

170. The figure for the '$A-B$' formulæ on page 119 is the same for all cases in which A and B are each less than 90°.

The figure given below is for the proof of the '$A+B$' formulæ, when, A and B being each less than 90°, their sum is greater than 90°.



The **words** of the proof are precisely those of page 118.

We may notice however that

$$\cos(A+B) = \frac{OM}{OP}\left[= \frac{-MO}{OP}\right] = \frac{-MK+OK}{OP} = \frac{OK}{OP} - \frac{MK}{OP},$$

and the rest follows as on page 118.

171. Thus we have proved that the 'A, B' formulæ are true provided A and B each lie between 0° and 90°.

The student can prove them for any other values by drawing the proper figure.

The 'A, B' formulæ are therefore true for any values whatever of the angles A and B.

172. By the aid of the 'A, B' formulæ we can prove the formulæ of Art. 140.

Example. Prove that $\sin(90^\circ + A) = \cos A$.

$$\begin{aligned}\sin(90^\circ + A) &= \sin 90^\circ \cos A + \cos 90^\circ \sin A,\\ &= 1 \times \cos A + 0 \times \sin A,\\ &= \cos A. \quad \text{Q. E. D.}\end{aligned}$$

EXAMPLES. XLIV.

Draw the figures for the first four of the following examples.

(1) For the $(A+B)$ formulæ, when A is greater than 90° and $(A+B)$ less than 180°.

(2) For the $(A-B)$ formulæ, when A and B each lie between 90° and 180°.

(3) For the $(A+B)$ formulæ, when A lies between 90° and 180°, and $(A+B)$ lies between 180° and 270°.

(4) For the $(A-B)$ formulæ, when A lies between 180° and 270°, and $(A-B)$ lies between 180° and A.

Deduce the six following formulæ from the 'A, B' formulæ.

(5) $\cos(90^\circ + A) = -\sin A$. (6) $\sin(90^\circ - A) = \cos A$.

(7) $\cos(90^\circ - A) = \sin A$. (8) $\sin(180^\circ - A) = \sin A$.

(9) $\cos(180^\circ - A) = -\cos A$. (10) $\sin(180^\circ + A) = -\sin A$.

(11) Assuming that the formula $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$ is true for all values of A and B, deduce the rest of the 'A, B' formulæ by the aid of the results on p. 107.

173. We may also **conversely** prove that the 'A, B' formulæ are true for angles of any magnitude by the aid of the result of page 107. (This is a very convenient method of proving the 'A, B' formulæ to be true for all values of the angles.)

For, assuming that the 'A, B' formulæ are true for certain values of the angles A and B, we can show that they are true if either of the angles A or B be *increased* by 90°.

Example.

$$\begin{aligned}\sin(90^\circ+A+B)&=\cos(A+B) \qquad \text{[p. 107.]}\\ &=\cos A\,.\cos B-\sin A\,.\sin B,\\ &=\sin(90^\circ+A)\,.\cos B-\{-\cos(90^\circ+A)\}\sin B,\\ &=\sin(90^\circ+A)\,.\cos B+\cos(90^\circ+A)\,.\sin B.\end{aligned}$$

Or, writing A' for $90^\circ+A$, we have

$$\sin(A'+B)=\sin A'\,.\cos B+\cos A'\,.\sin B.$$

We have proved (Art. 170) that the 'A, B' formulæ are true for all values of A and B between 0° and 90°. And therefore, by what we have said above, they are true for all values of A or B between 0° and 180°. And so on.

Therefore the 'A, B' formulæ are true for any values whatever of the angles A and B.

174. It follows that all formulæ deduced from the 'A, B' formulæ are true for angles of any magnitude whatever.

Thus the 'S, T' formulæ (page 126) are true for angles of any magnitude. Also the formulæ of the last Chapter for multiple angles, and all general formulæ in the Examples, are true for angles of any magnitude.

EXAMPLES. XLV.

(1) Deduce the values of sin 180° and cos 180° from those of the sine and cosine of 90°.

(2) The angle A is greater than 180° and less than 270° and $\tan A = \frac{1}{2}$: find $\sin 2A$ and $\sin 3A$.

(3) The angle θ lies in the fourth quadrant and $\cos\theta = \frac{1}{4}$, find $\sin 2\theta$ and $\sin 3\theta$. Find also $\cos 3\theta$, and hence determine in which quadrant 3θ lies.

(4) Prove that the different values of θ which satisfy the equation $\cos p\theta + \cos q\theta = 0$, form two series in *A. P.* with common differences $\frac{2\pi}{p+q}$ and $\frac{2\pi}{p \sim q}$ respectively.

ON SUBMULTIPLE ANGLES.

175. We have now proved all the formulæ of the last two Chapters to be universally true.

We may expect therefore that any result, which can be obtained from these formulæ by algebraical transformation, will have a complete geometrical interpretation. [See p. 116.]

176. Since, $$\cos A = 1 - 2\sin^2\frac{A}{2}, \qquad \text{[Art. 166.]}$$

and, $$\cos A = 2\cos^2\frac{A}{2} - 1;$$

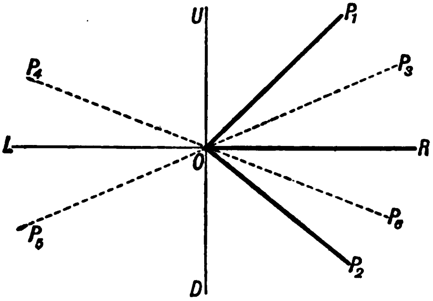
we have, $$\sin^2\frac{A}{2} = \frac{1-\cos A}{2},$$

and, $$\cos^2\frac{A}{2} = \frac{1+\cos A}{2}.$$

Or,
$$\left.\begin{aligned}\sin\frac{A}{2} &= \pm\sqrt{\frac{1-\cos A}{2}}\\ \cos\frac{A}{2} &= \pm\sqrt{\frac{1+\cos A}{2}}\end{aligned}\right\}.$$

Thus, given the value of $\cos A$ (nothing else being known about the angle A), we get **two** values for $\sin\frac{A}{2}$, one positive and one negative, and **two** like values for $\cos\frac{A}{2}$.

177. *To prove geometrically that, given the value of* $\cos A$ (nothing else being known about the angle A), *there are* **two** *values each of* $\sin\frac{A}{2}$ *and of* $\cos\frac{A}{2}$.



Let α be the least positive angle which has the given cosine, and let ROP_1 and P_2OR in the figure each $= \alpha$. Then A is one of the angles described by the revolving line OP when, starting from OR, OP stops either in the position OP_1 or in the position OP_2; i.e. any one of the angles

$$2n\pi \pm \alpha. \qquad \text{[Art. 147.]}$$

Taking the halves of these angles in succession, we have the four dotted lines OP_3, OP_4, OP_5, OP_6 in the figure, as the lines indicating the various values of the angle $\frac{A}{2}$; i.e. any one of the angles $n\pi \pm \frac{a}{2}$.

And it will be seen that

$$\sin ROP_3 = \sin ROP_4 = -\sin ROP_5 = -\sin ROP_6.$$

Also, $\cos ROP_3 = -\cos ROP_4 = -\cos ROP_5 = \cos ROP_6$.

From these it is clear that there are **two** values of $\sin \frac{A}{2}$, equal in magnitude and opposite in sign. Also, that there are **two** like values for $\cos \frac{A}{2}$.

EXAMPLES. XLVI.

(1) When A lies between -180^0 and 180^0, prove that

$$\cos \frac{A}{2} = +\sqrt{\frac{1+\cos A}{2}}.$$

(2) When A lies between 180^0 and 540^0, prove that

$$\cos \frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}.$$

(3) Find $\sin \frac{A}{2}$ in terms of $\cos A$, when A lies between 180^0 and 360^0.

(4) Prove that when A lies between $(4n+1)\pi$ and $(4n+3)\pi$, n being a positive integer, $\cos \frac{A}{2} = -\sqrt{\frac{1+\cos A}{2}}$.

(5) Find $\sin \frac{A}{2}$ in terms of $\cos A$, when A lies between $4n\pi$ and $(4n+2)\pi$, where n is a positive integer.

178. Since, $2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=\sin A$, [Art. 166.]

and, $\sin^2\frac{A}{2}+\cos^2\frac{A}{2}=1$,

we obtain by addition and subtraction

$$\sin^2\frac{A}{2}+\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=1+\sin A,$$

$$\sin^2\frac{A}{2}+\cos^2\frac{A}{2}-2\sin\frac{A}{2}\cdot\cos\frac{A}{2}=1-\sin A.$$

Therefore $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2=1+\sin A$,

and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A$.

Whence,

$$\left.\begin{aligned}\sin\frac{A}{2}+\cos\frac{A}{2}&=\pm\sqrt{1+\sin A}\\ \sin\frac{A}{2}-\cos\frac{A}{2}&=\pm\sqrt{1-\sin A}\end{aligned}\right\}\begin{aligned}&\text{......(i)},\\&\text{......(ii)}.\end{aligned}$$

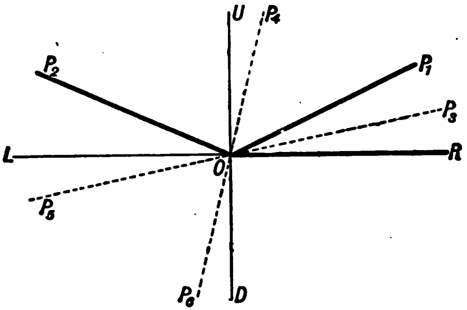
Adding we get, $2\sin\frac{A}{2}=\pm\sqrt{1+\sin A}\pm\sqrt{1-\sin A}$ (iii),

Subtracting we get, $2\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\mp\sqrt{1-\sin A}$ (iv).

Thus if we are given the value of $\sin A$, (nothing else being known about the angle A), we have **four** values for $\sin\frac{A}{2}$ and **four** values for $\cos\frac{A}{2}$.

179. *To prove Geometrically that, given the value of* $\sin A$ (nothing else being known about the angle A), *there are* **four** *values for* $\sin\frac{A}{2}$, *and* **four** *values for* $\cos\frac{A}{2}$.

Let a be the least positive angle which has the given sine, and let ROP_1, P_2OL in the figure each $= a$.



Then A is one of the angles described by the revolving line OP, when, starting from OR, OP stops either in the position OP_1 or in the position OP_2; i.e. any one of the angles

$$n\pi + (-1)^n \alpha. \qquad \text{[Art. 145.]}$$

Taking the halves of these angles in succession, we have the four dotted lines OP_3, OP_4, OP_5, OP_6 in the figure, as the lines indicating the various values of the angle $\frac{A}{2}$;

i.e. any one of the angles $\frac{n\pi}{2} + (-1)^n \frac{\alpha}{2}$.

In the figure, $ROP_3 = P_4OU = LOP_5 = P_6OD$.

And $\sin\frac{A}{2}$ may have any one of the values $\sin ROP_3$, $\sin ROP_4$, $\sin ROP_5$, $\sin ROP_6$. These values are all different, and are those given by the solution (iii).

Hence, there are **four** values for $\sin\frac{A}{2}$, of the nature indicated by the solution (iii); also **four** values for $\cos\frac{A}{2}$ (iv).

180. If we know the magnitude of A, we can decide which sign to take in the formulæ

$$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{1+\sin A}\ldots\ldots\ldots\ldots\text{(i)},$$

$$\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{1-\sin A}\ldots\ldots\ldots\ldots\text{(ii)}.$$

Example. When $\frac{A}{2}$ lies between -45^0 and $+45^0$,

$\cos\frac{A}{2}$ is *greater* than $\sin\frac{A}{2}$ and is *positive.*

So that $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ is *positive*, and $=+\sqrt{1+\sin A}$,

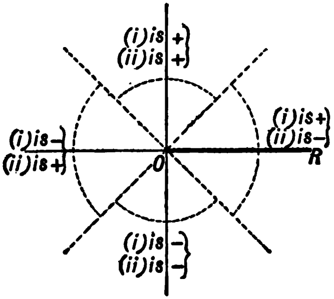
and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ is *negative*, and $=-\sqrt{1-\sin A}$.

When $\frac{A}{2}$ lies between $+45^0$ and $+135^0$,

$\sin\frac{A}{2}$ is greater than $\cos\frac{A}{2}$, and is positive.

So that $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ are both positive. And so on.

The following diagram completes the above results.



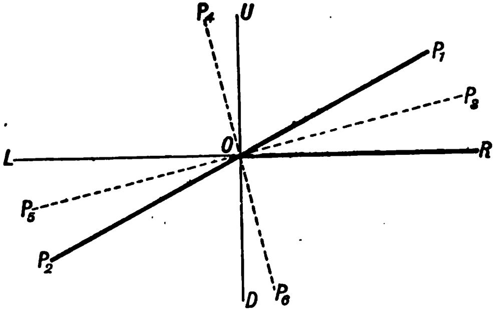
181. Since $$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}},$$

$$\therefore \tan^2\frac{A}{2} + \frac{2\tan\frac{A}{2}}{\tan A} - 1 = 0,$$

whence $$\tan\frac{A}{2} = \frac{-1 \pm \sqrt{1+\tan^2 A}}{\tan A}.$$

Thus, given $\tan A$ we find **two** unequal values for $\tan\frac{A}{2}$, one positive and one negative.

182. The student will be able by the aid of the following figure to verify this result geometrically.



183. We may remark that in this figure P_3OP_5 and P_4OP_6 are straight lines at right angles to each other. So that $\tan P_3OR = -\cot P_4OR$; or, $\tan P_3OR \, . \tan P_4OR = -1$.

Hence, one value of $\tan\frac{A}{2}$ is the reciprocal of the other, and of opposite sign. So that there is always one *positive* value of $\tan\frac{A}{2}$, and one *negative;* one numerically *greater* than unity and the other numerically *less* than unity.

Example. *If* $A=190^\circ$, *prove that*

$$\tan\frac{A}{2}=\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}.$$

Now $\tan A=\tan 190^\circ$, which is positive; $\tan\frac{A}{2}=\tan 95^\circ$, which is negative. Also $\tan\frac{A}{2}=\frac{-1\pm\sqrt{1+\tan^2 A}}{\tan A}$; the negative value of which is $\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}$.

EXAMPLES. XLVII.

(1) State the signs of $\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)$ and $\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)$ when $\frac{A}{2}$ has the following values:

(i) 22°, (ii) 191°, (iii) 290°, (iv) 345°,
(v) -22°, (vi) -275°, (vii) -470°, (viii) 1000°.

(2) Prove that the formulæ which give the values of $\sin\frac{A}{2}$ and of $\cos\frac{A}{2}$ in terms of $\sin A$, are unaltered when A has the values

(i) 92°, 268°, 900°, $4n\pi+\frac{3}{4}\pi$, or $(4n+2)\pi-\frac{3}{4}\pi$.

(ii) 88°, -88°, 770°, -770°, or $4n\pi\pm\frac{\pi}{8}$.

(3) Find the values of (i) $\sin 9^\circ$, (ii) $\cos 9^\circ$, (iii) $\sin 81^\circ$, (iv) $\cos 189^\circ$, (v) $\tan 202\frac{1}{2}^\circ$, (vi) $\tan 97\frac{1}{2}^\circ$.

(4) If $A=200^\circ$, prove that

(i) $$2\sin\frac{A}{2}=+\sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

(ii) $$\tan\frac{A}{2}=\frac{-(1+\sqrt{1+\tan^2 A})}{\tan A}.$$

(5) If A lie between 270° and 360°, prove that

(i) $$2\sin\frac{A}{2}=\sqrt{1-\sin A}-\sqrt{1+\sin A}.$$

(ii) $$\tan\frac{A}{2}=-\cot A+\operatorname{cosec} A.$$

(6) If A lie between 450° and 630° prove that

$$2\sin\frac{A}{2}=-\sqrt{1+\sin A}-\sqrt{1-\sin A}.$$

(7) Find the limits between which $\frac{A}{2}$ must lie when

$$2\sin\frac{A}{2}=\sqrt{1+\sin A}-\sqrt{1-\sin A},$$

(8) Given that A lies between 450^0 and 630^0, prove that

$$2\cos\frac{A}{2}=-\sqrt{1+\sin A}+\sqrt{1-\sin A}.$$

(9) If A lie between $n\times360^0-90^0$ and $n\times360^0+90^0$ where n is a positive integer, prove that

$$\tan\frac{A}{2}=\frac{-1+\sqrt{1+\tan^2 A}}{\tan A};$$

and that when A lies between $n\times360^0+90^0$ and $n\times360^0+270^0$, then

$$\tan\frac{A}{2}=\frac{-1-\sqrt{1+\tan^2 A}}{\tan A}.$$

(10) Prove geometrically that if we are given the value of $\sin A$, there are *three* different values for $\sin\frac{A}{3}$, and *six* different values for $\cos\frac{A}{3}$.

(11) Prove geometrically that, if we are given the value of $\cos A$, there are three different values for $\cos\frac{A}{3}$, and six different values for $\sin\frac{A}{3}$.

(12) Prove that if we are given the value of $\tan A$ there are three different values for $\tan\frac{A}{3}$.

(13) Given the value of $\tan A$, prove that there are *four* values each for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$.

(14) Given the value of $\sin A$, prove that there are *two* values for $\tan\frac{A}{2}$.

184. It is important if possible, in solving Trigonometrical equations, to avoid squaring both sides of the equation.

Example. *Solve* $\cos\theta = k\sin\theta$.

If we square both sides of the equation we get

$$\cos^2\theta = k^2\sin^2\theta = k^2(1-\cos^2\theta).$$

$$\therefore\ \cos^2\theta = \frac{k^2}{1+k^2},\ \text{or}\ \cos\theta = \pm\frac{k}{\sqrt{1+k^2}}.$$

Now if α be the least angle such that $\cos\theta = \dfrac{k}{\sqrt{1+k^2}}$,

then the above gives us $\theta = n\pi \pm \alpha$ ....(i).

But the equation may be written $\cot\theta = k$,

whence $\theta = n\pi + \alpha$ ....(ii).

(ii) is the complete solution of the proposed equation, while (i) is in fact the solution of both $\cos\theta = k\sin\theta$ and also of $\cos\theta = -k\sin\theta$. So that by squaring both sides of the equation we obtain solutions which do not belong to the given equation.

185. We can often avoid squaring by the use of a **Subsidiary** Angle.

Example. *Solve* $a\cos\theta + b\sin\theta = 1$.

That is $$a\left(\cos\theta + \frac{b}{a}\sin\theta\right) = 1.$$

Find in the tables the angle whose tangent is $\dfrac{b}{a}$; let it be α.

Then $\dfrac{b}{a} = \tan\alpha$; and the equation becomes

$$a(\cos\theta + \tan\alpha \,.\, \sin\theta) = 1,$$

or $$a\left(\frac{\cos\theta \,.\, \cos\alpha + \sin\theta \,.\, \sin\alpha}{\cos\alpha}\right) = 1,$$

or $$\cos(\theta - \alpha) = \frac{\cos\alpha}{a}.$$

Find from the Tables the value of $\cos a$. Next find from the Tables the magnitude of the angle β whose cosine $= \dfrac{\cos a}{a}$,

and we get
$$\cos(\theta - a) = \cos\beta;$$
$$\therefore\ \theta - a = 2n\pi \pm \beta,$$
or
$$\theta = a + 2n\pi \pm \beta.$$

186. **Def.** When an angle a is introduced to facilitate calculation it is called a **Subsidiary Angle.**

EXAMPLES. XLVIII.

Solve the following equations.

(1) $2\sin\theta + 2\cos\theta = \sqrt{2}$. (2) $\sin\theta + \sqrt{3}\,.\,\cos\theta = 1$.

(3) $\sqrt{2}\sin\theta + \sqrt{2}\cos\theta = \sqrt{3}$. (4) $\sin\theta - \cos\theta = 1$.

(5) $\sin\theta + \cos \quad = 1$. (6) $\sqrt{3}\sin\theta - \cos\theta \quad \sqrt{2} = 0$.

(7) $2\sin x + 5\cos x = 2$. $[2\cdot5 = \tan 68^0\,12']$

(8) $3\cos x - 8\sin x = 3$. $[2\cdot\dot{6} = \tan 69^0\,26'\,30'']$

(9) $4\sin x - 15\cos x = 4$. $[3\cdot75 = \tan 75^0\,4']$

(10) $\cos(a + x) = \sin(a + x) + \sqrt{2}\cos\beta$.

*On the Inverse Notation.

187. The equation $\sin\theta = a$ means that θ is an angle whose sine is a.

$\theta = \sin^{-1} a$ is a convenient way of writing the same equation.

Thus $\sin^{-1} a$ (is an **angle**, and) is an abbreviation for **an angle whose sine is** a.

Example 1. *Show that* 30^0 *is* one *value of* $\sin^{-1}\frac{1}{2}$.

We know that $\sin 30^0 = \frac{1}{2}$. Therefore 30^0 is an angle whose sine is $\frac{1}{2}$, or, $30^0 = \sin^{-1}\frac{1}{2}$.

Example 2. *Prove that* 45^0 *is one value of* $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

$\tan^{-1}\frac{1}{2}$ is one of the angles whose tangent is $\frac{1}{2}$.

Let $\alpha=\tan^{-1}\frac{1}{2}$, so that, $\tan\alpha=\frac{1}{2}$.

Let $\beta=\tan^{-}\frac{1}{3}$, so that, $\tan\beta=\frac{1}{3}$.

We have to prove that 45^0 is one value of $\alpha+\beta$.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,.\,\tan\beta}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\,.\,\frac{1}{3}}=1.$$

But $\tan 45^0=1$.

Therefore 45^0 is one value of $\alpha+\beta$, i.e. of $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$.

* EXAMPLES. XLIX.

Prove that the following statements are true when we take for $\sin^{-1}a$, etc. their least *positive* value.

(1) $\sin^{-1}\frac{3}{5}=\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$.

(2) $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}=\cot^{-1}\sqrt{3}$.

(3) $\sin^{-1}a=\cos^{-1}\sqrt{1-a^2}=\tan^{-1}\frac{a}{\sqrt{1-a^2}}$.

(4) If $\alpha=\sin^{-1}\frac{3}{5}$ and $\beta=\cos^{-1}\frac{3}{5}$, then $\alpha+\beta=\frac{\pi}{2}$.

(5) If $A=\sin^{-1}a$ and $B=\cos^{-1}a$, then $A+B=90^0$.

(6) $\tan^{-1}\frac{5}{7}+\tan^{-1}\frac{1}{6}=\frac{\pi}{4}$. (7) $\tan^{-1}\frac{2}{11}+2\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{1}{2}$.

(8) $\tan^{-1}m_1+\tan^{-1}m_2=\tan^{-1}\frac{m_1+m_2}{1-m_1m_2}$.

(9) $\sin(2\sin^{-1}a)=2a\sqrt{1-a^2}$.

(10) $2\cos^{-1}a=\cos^{-1}(2a^2-1)$.

(11) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}=120^0$. (12) $2\sin^{-1}\frac{4}{5}-\sin^{-1}\frac{24}{25}=2\cos^{-1}\frac{24}{25}$.

(13) $2\tan^{-1}(\cos 2a)=\tan^{-1}\left(\frac{\cot^2 a-\tan^2 a}{2}\right)$.

(14) $\tan^{-1}x+\tan^{-1}y+\tan^{-1}\frac{1-x-y-xy}{1+x+y-xy}=\frac{\pi}{4}$.

(15) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$.

(16) $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{13}{85}=\frac{\pi}{2}$.

(17) $\tan^{-1}\sqrt{5}\,(2-\sqrt{3})-\cot^{-1}\sqrt{5}\,(2+\sqrt{3})=\cot^{-1}\sqrt{5}$.

(18) If $$\sin^{-1}m+\sin^{-1}n=\frac{\pi}{2},$$
prove that $$m\sqrt{1-n^2}+n\sqrt{1-m^2}=1.$$

188. The student must notice carefully that such a statement as $\sin^{-1}\frac{1}{2}=\cos^{-1}\frac{\sqrt{3}}{2}$ is not an identity.

For $\sin^{-1}\frac{1}{2}$ is one of the values of $n\pi+(-1)^n\,30^\circ$,

and $\cos^{-1}\frac{\sqrt{3}}{2}$ is one of the values of $2n\pi\pm 30^\circ$.

Thus $150^\circ=\sin^{-1}\frac{1}{2}$, but 150° is not $=\cos^{-1}\frac{\sqrt{3}}{2}$.

**MISCELLANEOUS EXAMPLES. L.

(1) Prove that $\tan^{-1}\dfrac{1}{1+a}+\tan^{-1}\dfrac{1}{1-a}+\tan^{-1}\dfrac{2}{a^2}=n\pi$.

(2) Prove that $\tan^{-1}\dfrac{a-1}{a}+\tan^{-1}\dfrac{1}{2a-1}=n\pi+\dfrac{\pi}{4}$.

(3) Prove that
$$\sin^{-1}x-\sin^{-1}y=\cos^{-1}\{xy\pm\sqrt{1-x^2-y^2+x^2y^2}\}.$$

(4) If $\tan^{-1}\dfrac{x+1}{x+2}+\tan^{-1}\dfrac{x-1}{x-2}=\dfrac{\pi}{4}$, show that $x^2=\frac{1}{2}$

(5) Prove that $\tan^{-1}a+\cot^{-1}a=(2n+1)\dfrac{\pi}{2}$.

(6) If $$\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=\pi,$$
prove that $$\alpha+\beta+\gamma=\alpha\beta\gamma.$$

(7) Solve the equation $\tan^{-1}\dfrac{1}{x-1}-\tan^{-1}\dfrac{1}{x+1}=\dfrac{\pi}{12}$.

(8) Solve the equation

$$\tan^{-1}(x+1)-\tan^{-1}(x-1)=\cot^{-1}(x^2-1).$$

(9) Solve the equation $\sin^{-1}\dfrac{2x}{1+x^2}+\tan^{-1}\dfrac{2x}{1-x^2}=\dfrac{\pi}{2}.$

(10) Prove that

$$\tan^{-1}\frac{a-\sqrt{a^2-4}}{2\sqrt{a+1}}+\tan^{-1}\frac{1}{\sqrt{a+1}}+\tan^{-1}\frac{a+\sqrt{a^2-4}}{2\sqrt{a+1}}=n\pi+\frac{\pi}{2}.$$

(11) If a be positive and less than unity, and if α be the least value of $\sin^{-1}a$, then

$$\sin^{-1}a+\cos^{-1}a=n\pi+(-1)^n\alpha\pm\left(\frac{\pi}{2}-\alpha\right).$$

(12) Prove that $\tan A$ and $\sin 2A$ have always the same sign.

Solve the six following equations.

(13) $\cos A+\cos 3A+\cos 5A=0.$

(14) $\sin 5\theta+\sin 3\theta+\sin\theta=3-4\sin^2\theta.$

(15) $2\sin^2 3A+\sin^2 6A=2.$

(16) $a(\cos 2x-1)+2b(\cos x+1)=0.$

(17) $\sin(m+n)\theta+\sin 2m\theta+\sin(m-n)\theta=0.$

(18) $\left.\begin{array}{c}\sin\{\pi x(x+y)\}+\sin\{\pi y(x+y)\}=0,\\ \sin\pi x^2+\sin\pi y^2=0.\end{array}\right\}$

(19) Trace the changes in the sign and magnitude of the following expressions, as θ changes from 0 to π.

(i) $2\sin\theta\,.\,\cos\theta.$ (ii) $\cos^2\theta-\sin^2\theta.$
(iii) $\sin 3\theta.$ (iv) $\cot 2A.$
(v) $\sin(\theta+\alpha).$ (vi) $\cos(2\theta-\alpha).$

(20) Explain why the equations

$$\theta=n\pi+(-1)^n\alpha \quad\text{and}\quad \frac{\pi}{2}-\theta=2n\pi\pm\left(\frac{\pi}{2}-\alpha\right)$$

have exactly the same series of solutions.

(21) Explain why exactly the same series of angles are given by the two equations $\theta+\dfrac{\pi}{4}=n\pi+(-1)^n\dfrac{\pi}{6}$, and $\theta-\dfrac{\pi}{4}=2n\pi\pm\dfrac{\pi}{3}.$

CHAPTER XIV.

ON LOGARITHMS.

189. IN Algebra it is explained

(i) that the *multiplication* of different powers of the same quantity is effected by *adding* the *indices* of those powers;

(ii) that *division* is effected by *subtracting* the *indices;*

(iii) that *involution* and *evolution* are respectively effected by the *multiplication* and *division* of the *indices.*

Example 1. If $m=a^h$, $n=a^k$,

then
$$m\times n=a^h\times a^k=a^{h+k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$
$$m\div n=a^h\div a^k=a^{h-k} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$
$$\left.\begin{array}{l} m^3=(a^h)^3=a^{3h}, \\ \sqrt[4]{m}=m^{\frac{1}{4}}=(a^h)^{\frac{1}{4}}=a^{\frac{h}{4}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Example 2. If $347=10^{2\cdot5403295}$* and $461=10^{2\cdot6637009}$, prove that
$$347\times461=10^{5\cdot2040304}.$$

We have
$$347\times461=10^{2\cdot5403295}\times10^{2\cdot6637009}$$
$$=10^{2\cdot5403295+2\cdot6637009}$$
$$=10^{5\cdot2040304}. \qquad \text{Q. E. D.}$$

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* The number 347 lies between 100 and 1000, i. e. between 10^2 and 10^3. Hence, if there is a power of 10 which is equal to 347, its index must be greater than 2 and less than 3, i. e. equal to 2 + a fraction.

EXAMPLES. LI.

(1) If $m=a^h$, $n=a^k$, express in terms of a, h and k,

(i) $m^2 \times n^3$. (ii) $m^4 \div n^5$. (iii) $\sqrt[3]{m^4 \times n^5}$. (iv) $\{\sqrt[4]{m^5 \times n^3}\}^2$.

(2) If $453=10^{2·6560982}$ and $650=10^{2·8129134}$, find the indices of the powers of 10 which are equal to

(i) 453×650. (ii) $(453)^4$. (iii) $650^3 \times 453^2$. (iv) $\sqrt[3]{453}$.

(v) $\sqrt{453} \times \sqrt[6]{650}$. (vi) $\sqrt[5]{453} \times (650)^3$. (vii) $\sqrt{453 \times 650}$.

(3) Express in powers of 2 the numbers, 8, 32, $\frac{1}{2}$, $\frac{1}{16}$, ·125, 128.

(4) Express in powers of 3 the numbers, 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, ·$\dot{1}$, $\frac{1}{81}$.

190. Suppose that some convenient number (such as 10) having been chosen, we are given a list of the indices of the powers of that number, which are equivalent to every whole number from 1 up to 100000

Such a list could be used to shorten Arithmetical calculations.

Example 1. Multiply 3759 by 4781 and divide the result by 2690

Looking in our *list* we should find $3759=10^{3·5750723}$, $4781=10^{3·6795187}$, $2690=10^{3·4297523}$.

Therefore $3759 \times 4781 \div 2690 = 10^{3·5750723} \times 10^{3·0795187} \div 10^{3·4297523}$
$= 10^{3·5750723+3·0795187-3·4297523} = 10^{3·8248387}$.

The list will give us that $10^{3·8248387}=6680·9$.

Therefore the answer correct to five significant figures is 6680·9.

Example 2. Simplify $3^6 \times 2^{10} \div \sqrt[3]{17601}$.

The *list* gives $2=10^{·3010300}$, $3=10^{·4771213}$ and $17601=10^{4·2455373}$.

Thus $3^6 \times 2^{10} \div \sqrt[3]{17601} = (10^{\cdot 4771213})^6 \times (10^{\cdot 3010300})^{10} \div (10^{4\cdot 2455373})^{\frac{1}{3}}$

$= 10^{2\cdot 8627278} \times 10^{3\cdot 0103000} \div 10^{1\cdot 4151791}$

$= 10^{2\cdot 8627278 + 3\cdot 0103000 - 1\cdot 4151791}$

$= 10^{4\cdot 4578487}$.

And from our list we find $10^{4\cdot 4578487} = 28697$, nearly.

EXAMPLES. LII.

Given that $2 = 10^{\cdot 3010300}$, $3 = 10^{\cdot 4771213}$ and $7 = 10^{\cdot 8450980}$, find the indices of the powers of 10 equivalent to the quantities in the first 6 examples.

(1) $2^2, 3^2, 2^3, 2 \times 3, 2^4, 7^2$.

(2) 14, 16, 18, 24, 27, 42.

(3) 10, 5, 15, 25, 30, 35.

(4) 36, 40, 48, 50, 200, 1000.

(5) $3^{10} \times 7^{10} \div 2^{20}$, $2^{12} \times 3^{20} \div 7^{11}$.

(6) $\sqrt[3]{21} \times \sqrt[4]{18}$, $\sqrt[2]{\cdot 49 \times 4^5} \times \sqrt[3]{3^4 \times 2^{10}}$.

(7) Find approximately the numerical value of $\sqrt[10]{42}$ having given that $10^{\cdot 1623249} = 1\cdot 4532$ nearly.

(8) Find approximately the numerical value of $\sqrt[3]{(42)^4} \times \sqrt[4]{(42)^3}$ having given that $10^{3\cdot 38177} = 2408\cdot 6$.

(9) Find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$, (ii) of $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$ having given that $10^{\cdot 6615067} = 4\cdot 5868$ and $10^{-\cdot 0285094} = \cdot 93646$.

(10) Find the value of $(67\cdot 21)^{\frac{3}{5}} \times (49\cdot 62)^{\frac{1}{5}} \times (3\cdot 971)^{-\frac{7}{5}}$ having given that $67\cdot 21 = 10^{1\cdot 8274339}$, $49\cdot 62 = 10^{1\cdot 6956568}$, $3\cdot 971 = 10^{\cdot 5988999}$ and $10^{\cdot 5971310} = 3\cdot 9549$.

(11) Find the area of a square field whose side is 640·12 feet; having given that $640\cdot 12 = 10^{2\cdot 8062614}$ and that $10^{5\cdot 6125228} = 40975\cdot 3$.

(12) Find the edge of a solid cube which contains 42601 cubic inches; having given $42601 = 10^{4\cdot 6294198}$ and $10^{1\cdot 5431399} = 34\cdot 925$.

(13) Find the edge of a solid cube which contains 34·701 cubic inches; having given that $34\cdot 701 = 10^{1\cdot 5403420}$, and $10^{\cdot 5134473} = 3\cdot 2617$.

(14) Find the volume of the cube the length of one of whose edges is 47·931 yards; having given that $47\cdot 931 = 10^{1\cdot 6806165}$ and that $10^{5\cdot 0418495} = 110115$.

191. The powers of any other number than 10 might be used in the manner explained above, but 10 is the most *convenient* number, as will presently appear.

192. This method, in which the *indices* of the powers of a certain fixed number (such as 10) are made use of, is called the *Method of Logarithms.*

Indices thus used are called **logarithms.**

The *fixed number* whose powers are used is called the **base.** Hence we have the following definition:

DEF. The **logarithm** of a number to a given *base*, is the **index** of that power of the base, which is equal to the given number.

Thus, if l be the logarithm of the number n to the base a, then

$$a^l = n.$$

193. The **notation** used is $\log_a n = l.$

Here, $\log_a n$ is an abbreviation for the words 'the logarithm of the number n to the base a.' And this means, as we have explained above, 'the **index** of that power of a which is equal to the number n.'

Example 1. What is the logarithm of $a^{\frac{3}{2}}$ to the base a?

That is, what is the **index** of the power of a which is $a^{\frac{3}{2}}$?

The index is $\frac{3}{2}$; therefore $\frac{3}{2}$ is the required logarithm, or

$$\log_a a^{\frac{3}{2}} = \frac{3}{2}.$$

Example 2. What is the logarithm of 32 to the base 2?

That is, what is the **index** of the power of 2 which is equal to 32?

Now $32 = 2^5$. ∴ the required index is 5; or $\log_2 32 = 5$.

Example 3. Given that $\log_{10} 2 = \cdot 3010300$, find $\log_{10} 8$ and $\log_{10} 2000$.

That $\log_{10} 2 = \cdot 3010300$, means that $10^{\cdot 3010300} = 2$,

and $$8 = 2^3 = (10^{\cdot 3010300})^3 = 10^{\cdot 9030900},$$

$$\therefore\ 8 = 10^{\cdot 9030900}, \text{ or } \log_{10} 8 = \cdot 9030900.$$

Again $$2000 = 2 \times 10^3 = 10^{\cdot 3010300} \times 10^3$$
$$= 10^{\cdot 3010300 + 3} = 10^{3 \cdot 3010300};$$
$$\therefore\ \log_{10} 2000 = 3 \cdot 3010300.$$

EXAMPLES. LIII.

(1) Find the logarithms to the base a of a^3, $a^{\frac{1}{3}}$, $\sqrt[4]{a}$, $\sqrt[3]{a^2}$, $\frac{1}{a^{\frac{5}{4}}}$.

(2) Find the logarithms to the base 2 of 8, 64, $\frac{1}{2}$, $\cdot 125$, $\cdot 015625$, $\sqrt[3]{64}$.

(3) Find the logarithms to the base 3 of 9, 81, $\frac{1}{3}$, $\frac{1}{27}$, $\cdot\dot{1}$, $\frac{1}{81}$.

(4) Find the logarithms to base 4 of 8, $\sqrt[3]{16}$, $\sqrt[4]{\cdot 5}$, $\sqrt[3]{\cdot 015625}$.

(5) Find the value of
$\log_2 8$, $\log_2 \cdot 5$, $\log_3 243$, $\log_5 (\cdot 04)$, $\log_{10} 1000$, $\log_{10} \cdot 001$.

(6) Find the value of
$\log_a a^{\frac{4}{3}}$, $\log_b \sqrt[3]{b^2}$, $\log_8 2$, $\log_{27} 3$, $\log_{100} 10$.

If $2 = 10^{\cdot 3010300}$, $3 = 10^{\cdot 4771213}$ and $7 = 10^{\cdot 8450980}$, find the values of

(7) $\log_{10} 6$, $\log_{10} 42$, $\log_{10} 16$. (8) $\log_{10} 49$, $\log_{10} 36$, $\log_{10} 63$.

(9) $\log_{10} 200$, $\log_{10} 600$, $\log_{10} 70$. (10) $\log_{10} 5$, $\log_{10} 3\cdot\dot{3}$, $\log_{10} 50$.

(11) $\log_{10} 35$, $\log_{10} 150$, $\log_{10} \cdot 2$. (12) $\log_{10} 3\cdot 5$, $\log_{10} 7\cdot 29$, $\log_{10} \cdot 081$.

(13) Given $\log_{10} 2$, $\log_{10} 3$, $\log_{10} 7$, find the value (i) of $\sqrt[3]{6} \times \sqrt[4]{7} \times \sqrt[5]{9}$, (ii) of $\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}$
$[\cdot 6615067 = \log_{10} 4\cdot 5868;\ -\cdot 0285094 = \log_{10} \cdot 93646]$.

(14) Prove that (i) $\log\{\sqrt[3]{2} \times \sqrt[4]{7} \div \sqrt[5]{9}\} = \frac{1}{3}\log 2 + \frac{1}{4}\log 7 - \frac{2}{5}\log 3$,
(ii) $\log\{\sqrt[10]{2} \times 3^{-\frac{5}{4}} \times 7^{\frac{7}{11}}\} = \frac{1}{10}\log 2 - \frac{5}{4}\log 3 + \frac{7}{11}\log 7$.

COMMON LOGARITHMS.

194. That System of Logarithms whose base is 10, is called the Common System of Logarithms.

In speaking of logarithms hereafter, *common* logarithms are referred to unless the contrary is expressly stated.

195. We shall assume that a power of ten can be found which is practically equivalent to any number.

196. The indices of these powers of 10, *i.e.* the Common Logarithms, are in general *incommensurable* numbers.

Their value for every whole number, from 1 to 100000, has been calculated to 7 significant figures. Thus any calculation made with the aid of logarithms is as exact as the most carefully observed measurement (cf. Arts. 17, 216).

197. Now, the greater the index of any power of 10, the greater will be the numerical value of that power; and the less the index, the less will be the numerical value of the power.

Hence, if one number be less than another, the logarithm of the first will be less than the logarithm of the second.

But the student should notice that logarithms (or indices) are *not proportional* to the corresponding numbers.

Example. 1000 is less than 10000; and the logarithm to base 10 of the first is 3 and of the second is 4

But 1000, 10000, 3, 4 are not in proportion.

198. We know from Algebra that

$$1 = 10^0,$$

$$\begin{array}{rcr} 10 = 10^1 & \text{and that} & \cdot 1 = \frac{1}{10} = 10^{-1} \\ 100 = 10^2 & \ldots\ldots\ldots & \cdot 01 = \frac{1}{100} = 10^{-2} \\ 1000 = 10^3 & \ldots\ldots\ldots & \cdot 001 = \frac{1}{1000} = 10^{-3} \\ 10000 = 10^4 & \ldots\ldots\ldots & \cdot 0001 = \frac{1}{10000} = 10^{-4} \end{array}$$

and so on.

Hence, the logarithm of 1 is 0.

The (common) logarithm of any number greater than 1 is *positive.*

The logarithm of any positive number less than 1 is *negative.*

199. We observe also

that the logarithm of any number between 1 and 10 is a positive decimal fraction;

that the logarithm of any number between 10 and 100, *i.e.* between 10^1 and 10^2, is of the form 1 + a decimal fraction;

that the logarithm of any number between 1000 and 10000, *i.e.* between 10^3 and 10^4, is of the form 3 + a decimal fraction;

and so on.

200. We observe also

that the logarithm of any number between 1 and ·1, *i.e.* between 10^0 and 10^{-1}, can be written in the form -1 + a decimal fraction;

that the logarithm of any number between ·1 and ·01, *i.e.* between 10^{-1} and 10^{-2}, can be written in the form -2 + a decimal fraction;

and so on.

Example 1. How many digits are contained in the integral part of the number whose logarithm is 3·67192?

The number is $10^{3·67192}$ and this is greater than 10^3, *i.e.* greater than 1000, and it is less than 10^4, *i.e.* less than 10000. Therefore the number lies between 1000 and 10000, and therefore the integral part of it contains 4 figures.

Example 2. Given that $3 = 10^{·4771213}$, find the number of the digits in the integral part of 3^{20}.

We have $$3 = 10^{·4771213},$$

$$\therefore\ 3^{20} = (10^{·4771213})^{20} = 10^{9·5424260}.$$

Therefore there are 10 digits in the integral part of 3^{20}; for it is greater than 10^9 and less than 10^{10}.

Example 3. Supposing that the decimal part of the logarithm is to be kept positive, find the integral part of the logarithm of ·0001234.

This number is greater than ·0001 *i.e.* than 10^{-4} and less than ·001, *i.e.* than 10^{-3}.

Therefore its logarithm lies between -3 and -4, and therefore it is $-4 +$ a fraction; the integral part is therefore -4.

EXAMPLES. LIV.

Note. The *decimal* part of a logarithm is to be kept *positive.*

(1) Write down the integral part of the common logarithms of 17601, 361·1, 4·01, 723000, 29.

(2) Write down the integral part of the common logarithms of ·04, ·0000612, ·7963, ·001201. (See Note above.)

(3) Write down the integral part of the common logarithms of 7963, ·1, 2·61, 79·6341, 1·0006, ·00000079.

(4) How many digits are there in the integral part of the numbers whose common logarithms are respectively

3·461, ·3020300, 5·4712301, 2·6710100?

(5) Give the position of the first *significant* figure in the numbers whose logarithms are

$$-2+{\cdot}4612310,\ -1+{\cdot}2793400,\ -6+{\cdot}1763241.$$

(6) Give the position of the first *significant* figure in the numbers whose common logarithms are $4{\cdot}2990713$, ${\cdot}3040595$, $2{\cdot}5860244$, $-3+{\cdot}1760913$, $-1+{\cdot}3180633$, ${\cdot}4980347$.

(7) Given that $2=10^{\cdot 3010300}$, find the number of digits in the integral part of 8^{10}, 2^{12}, 16^{20}, 2^{100}.

(8) Given that $\log 7={\cdot}8450980$, find the number of digits in the integral part of 7^{10}, 49^{6}, $343^{\frac{100}{3}}$, $(\frac{10}{7})^{20}$, $(4{\cdot}9)^{12}$, $(3{\cdot}43)^{10}$.

(9) Find the position of the first significant figure in

$$\sqrt[10]{2},\ (\tfrac{1}{2})^{10},\ (\tfrac{10}{7})^{20},\ ({\cdot}02)^{4},\ ({\cdot}49)^{6}.$$

(10) Find the position of the first significant figure in the numerical value of

$$20^{7},\ ({\cdot}02)^{7},\ ({\cdot}007)^{2},\ (3{\cdot}43)^{\frac{1}{10}},\ ({\cdot}0343)^{8},\ ({\cdot}0343)^{\frac{1}{10}}.$$

201. **Prop.** *To prove that if two numbers expressed in the decimal notation have the same digits (so that they differ only in the position of the decimal point), their logarithms to the base* 10 *will differ only by an integer.*

The decimal point in a number is moved by multiplying or dividing the number by some *integral* power of 10.

Let the logarithm to base 10 of the first number be l; so that the first number is 10^l. The second number is equal to the first number multiplied or divided by some *integral* power of 10 *i.e.* $=10^l \times 10^n$, where n is some integer, positive or negative.

Therefore the second number is 10^{l+n};

and therefore its logarithm is $l+n$,

which is the logarithm of the first number ± an integer.

Example. 17692 and 17·692 are numbers whose logarithms to the base 10 differ by the whole number 3:

for $$17692 = 10^{4\cdot247776},$$

and $$17\cdot692 = \frac{17692}{1000} = \frac{10^{4\cdot247776}}{10^3} = 10^{4\cdot247776\ldots-3},$$

and their logarithms are therefore 4·247776 and 1·247776 respectively.

202 It is *convenient* to keep the *decimal part* of common logarithms always *positive*, because then the *decimal part* of the logarithms of any numbers expressed by the same digits will be *always* the same.

203. The decimal part of a logarithm is called the **mantissa.**

204. The integral part is called the **characteristic.**

205. The characteristic of a logarithm can be always obtained by the following rule, which is evident from page 167.

RULE. The characteristic of the logarithm of a number greater than unity is **one less** than the number of *integral* figures in that number.

The characteristic of a number less than unity is negative, and (when the number is expressed as a decimal,) is **one more** than the number of cyphers between the decimal point and the first significant figure to the right of the decimal point.

206. When the characteristic is negative, as for example in the logarithm $-3 + \cdot1760913$, the logarithm is abbreviated thus, $\bar{3}\cdot1760913$.

Example 1. The characteristics of 36741, 36·741, ·0036741, 3·6741 and ·36741 are respectively 4, 1, −3, 0, and −1.

Example 2. Given that the mantissa of the logarithm of 36741 is 5651510, we can at once write down the logarithm of any number whose digits are 36741.

Thus

$$\begin{aligned} \log 3674100 &= 6{\cdot}5651510, \\ \log 36741 &= 4{\cdot}5651510, \\ \log 367{\cdot}41 &= 2{\cdot}5651510, \\ \log {\cdot}36741 &= \bar{1}{\cdot}5651510, \\ \log {\cdot}00036741 &= \bar{4}{\cdot}5651510, \end{aligned}$$

and so on.

207. In any set of tables of common logarithms the student will find the *mantissa only* corresponding to any set of digits.

It would obviously be superfluous to give the *characteristic.*

208. It is most important to remember to keep the **mantissa** always **positive.**

Example. Find the fifth root of ·00065061.

Here $\log_{10} {\cdot}00065061 = \bar{4}{\cdot}8133207,$

$$\therefore \log_{10} ({\cdot}00065061)^{\frac{1}{5}} = \tfrac{1}{5}(\bar{4}{\cdot}8133207) = \tfrac{1}{5}(-4 + {\cdot}8133207)$$

$$= \tfrac{1}{5}(-5 + 1{\cdot}8133207) = -1 + {\cdot}3626641 = \bar{1}{\cdot}3626641,$$

and $\bar{1}{\cdot}3626641 = \log {\cdot}23050,$

$\therefore$ the fifth root of ·00065062 = ·23050 nearly.

EXAMPLES. LV.

(1) Write down the logarithms of 776·43, 7·7643, ·00077643 and 776430. (The table gives opposite the numbers 77643, the figures 8901023.)

(2) Given that $\log_{10} 59082 = 4{\cdot}7714552$, write down the logarithms of 5908200, 5·9082, ·00059082, 590·82 and 5908·2.

(3) Find the fourth root of ·0059082, having given that

$$\log 5{\cdot}9082 = {\cdot}7714552\,; \quad 4{\cdot}4428638 = \log_{10} 27724.$$

(4) Find the product of ·00059082 and ·027724, having given that ·21431 = log 16380 (cf. Question 3).

(5) Find the 10th root of ·077643 (cf. Question 1), having given that ·8890102 = log 7·7448.

(6) Find the product of $({\cdot}27724)^2$ and ·077643. (See Questions 1 and 3; 7758288 = log 59680.)

* * 209. *To transform a system of logarithms having a given base, to another system with a different base.*

If we are given a list of logarithms calculated to a given base, we can deduce from it a list of logarithms calculated to any other base.

Let a be the given base; let b be any other base.

Let m be any number. Then the logarithm of m to the base a is in the given list. Let this logarithm be l. Then

$$m = a^l.$$

We wish to find the logarithm of m to the base b. Let it be x.

Then $$m = b^x. \quad \text{But } m = a^l;$$

$$\therefore\ a^l = b^x\,; \ \text{or, } b = a^{\frac{l}{x}};$$

or, $\dfrac{l}{x}$ is the logarithm of b to the base a.

Now the logarithm of b to the base a is given. For it is in the list of logarithms to the base a.

Thus, $$\frac{l}{x} = \log_a b\,; \ \text{or, } x = \frac{l}{\log_a b}\,;$$

or, $$\log_b m = \frac{\log_a m}{\log_a b}.$$

Hence, to calculate the logarithms of a series of numbers to a new base b, we have only to divide each of the logarithms of the numbers to any given base a, by a certain divisor, viz. $\log_a b$.

If then a list of logarithms to some base e can be made, we can deduce from it a list of common logarithms, by multiplying each logarithm in the given list by $\dfrac{1}{\log_e 10}$.

Example. Show how to transform logarithms, having 5 for base to logarithms having 125 for base.

Suppose $m = 5^l$, so that $l = \log_5 m$.

Now $125 = 5^3$, so that $3 = \log_5 125$,

$$\text{and} \quad m = 5^{3 \times \frac{l}{3}} = 125^{\frac{l}{3}}, \text{ so that } \frac{l}{3} = \log_{125} m.$$

Thus the logarithm of any number to base 5, divided by 3 (*i.e.* by $\log_5 125$), is the logarithm of the same number to the base 125.

**210. The student will find that the logarithms of numbers cannot be calculated to the base 10 directly.

They are first calculated to the base 2·7182818, etc., which is the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{etc. } \textit{ad inf.}$$

This number is called e.

And the constant divisor in this case is $\log_e 10$,

$$\text{and} \qquad \frac{1}{\log_e 10} = \frac{1}{2{\cdot}30258509} = {\cdot}43429448, \text{ etc.}$$

When this constant divisor is transformed into a *multiplier*, this constant **multiplier** is called a **modulus.**

Example. Given that $\log_{10} 12 = 1{\cdot}0791812$, shew how to transform common logarithms to logarithms having 12 for base.

Here $$10^{1\cdot 0791812} = 12;$$

$$\therefore 10 = 12^{\frac{1}{1\cdot 0791812}} = 12^{\cdot 92662,\text{ etc.}}$$

$$\therefore 10^h = 12^{h \times \cdot 92662}.$$

* * EXAMPLES. LVI.

(1) Show how to transform logarithms having 2 for base to logarithms having 8 for base.

(2) Show how to transform logarithms having 9 for base to logarithms having 3 for base.

(3) Show how to transform common logarithms to logarithms having 2 for base ($\log_{10} 2 = \cdot 3010300$).

(4) Show how to transform logarithms having 3 for base to common logarithms ($\log_{10} 3 = \cdot 4771213$).

(5) Show how to transform common logarithms to logarithms having 3 for base.

(6) Given $\log_{10} 2 = \cdot 3010300$, find $\log_2 10$.

(7) Given $\log_{10} 7 = \cdot 8450980$, find $\log_7 10$.

(8) Given $\log_{10} 2 = \cdot 3010300$, find $\log_8 10$ and $\log_{32} 10$.

*211. We give here a formal proof of the following propositions:

To prove that

(i) The logarithm of a *product* is equal to the *sum* of the logarithms of the factors.

(ii) The logarithm of a *quotient* is equal to the *difference* of the logarithms of the dividend and divisor

(iii) The logarithm of the *power* of a number is equal to the *product* of the logarithm of the number by the index denoting the power.

(iv) The logarithm of the *root* of a number is equal to the result of *dividing* the logarithm of the number by the number denoting the root.

Let m and n be any two numbers.

Let h and k be their logarithms to any base a.

So that $\log_a m = h$, $\log_a n = k$

and $\therefore m = a^h$, $n = a^k$.

Then

(i) $m \times n = a^h \times a^k = a^{h+k}$, $\therefore \log_a m \times n = h + k = \log_a m + \log_a n$.

(ii) $m \div n = a^h \div a^k = a^{h-k}$, $\therefore \log_a m \div n = h - k = \log_a m - \log_a n$.

(iii) $m^r = (a^h)^r = a^{rh}$, $\therefore \log_a m^r = rh = r \log_a m$.

(iv) $\sqrt[r]{m} = (a^h)^{\frac{1}{r}} = a^{\frac{h}{r}}$, $\therefore \log_a \sqrt[r]{m} = \frac{h}{r} = \frac{1}{r} \times \log_a m$.

*212. The above is only a formal way of saying, that since logarithms are **indices,** therefore they obey the *index law.*

* * MISCELLANEOUS EXAMPLES. LVII.

(1) Find $\log_2 8$, $\log_5 1$, $\log_8 2$, $\log_7 1$, $\log_{32} 128$.

(2) Show that the logarithms of all except eight of the numbers from 1 to 30 inclusive, can be calculated in terms of log 2, log 3 and log 7.

(3) Show that the logarithms of the numbers 1 to 10 inclusive may be found in terms of the logarithms of 8, 14, 21.

(4) The mantissa of the log of 85762 is 9332949. Find the log of $\sqrt[11]{\cdot 0085762}$.

Find how many figures there are in the integral part of $(85762)^{11}$.

(5) Find the product of 47·609, 476·09, ·47609, ·000047609, having given that log 4·7609 = ·6776891 and ·7107564 = log 5·1375.

(6) What are the characteristics of the logarithm of 3742 to the bases 3, 6, 10 and 12 respectively.

(7) Having given that log 2 = ·3010300, log 3 = ·4771213 and log 7 = ·8450980, solve the following equations:

(i) $2^x \times 3^{4x} = 7^2$, (ii) $3^{2x} = 128 \times 7^{4-x}$,

(iii) $12^x = 49$, (iv) $2^{8x} = 21^{4-8x}$.

(8) Given $\log_{10} 7$, find $\log_7 490$.

(9) Given $\log_{10} 3$, find $\log_9 270$.

(10) Given $\log_{10} 2$, find $\log_5 10$.

(11) Given $\log_8 9 = a$, $\log_2 5 = b$, $\log_5 7 = c$; find the logs to base 10 of numbers 1 to 7 inclusive.

(12) How many positive integers are there whose logarithms to base 2 have 5 for a characteristic?

(13) If a be an integer, how many positive integers are there whose logs to base a have 10 for their characteristic?

(14) Given log 2 and log 7, find the eleventh root of $(39{\cdot}2)^2$.

$$\log 1{\cdot}9485 = {\cdot}289688.$$

(15) Prove that $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$.

(16) Prove that

$$2 \log a + 2 \log a^2 + 2 \log a^3 \ldots + 2 \log a^n = n(n+1) \log a.$$

(17) Prove that $\log_a b \,.\, \log_b a = 1$; and that $\log_a b \,.\, \log_b c \,.\, \log_c a = 1$.

(18) Prove that $\log_a r = \log_a b \,.\, \log_b c \,.\, \log_c d \ldots \log_q r$.

(19) Given that the integral part of $(3{\cdot}456)^{100000}$ contains 53856 digits, find log 345·6 correct to five places of decimals.

(20) Given that the integral part of $(3{\cdot}981)^{100000}$ contains sixty thousand digits, find log 39810 correct to five places of decimals.

(21) If the number of births in a year be $\frac{1}{48}$ of the population at the beginning of the year, and the number of deaths $\frac{1}{60}$, find in what time the population will be doubled.

Given log 2, log 3, and that log 241 = 2·3820170.

(22) Prove that

$$\log s + \log (s-a) - \log b - \log c = 2 \log \sqrt{\frac{s(s-a)}{bc}}.$$

(23) Prove that

$$\log (a^2 + x^2) + \log (a+x) + \log (a-x) = \log (a^4 - x^4).$$

(24) Prove that

$$\log \sin 4A = \log 4 + \log \sin A + \log \cos A + \log \cos 2A.$$

CHAPTER XV.

ON THE USE OF MATHEMATICAL TABLES.

213. The Logarithms referred to in this chapter, and in future throughout the book, are *Common* Logarithms.

214. Books of Mathematical Tables usually give an explanation of their own contents, but there are some points common to all such Tables which we proceed to explain.

215. The student will be supposed to have access to a book containing the following:

(i) A list of the logarithms of all whole numbers from 1 to 99999, calculated to seven significant figures;

(ii) A list of the numerical values, calculated to seven significant figures, of the Trigonometrical Ratios of all angles, between 0° and 90°, which differ by 1′;

(iii) A list of the logarithms of these Ratios calculated to seven significant figures.

These will be found in Chambers' *Mathematical Tables*.

216. We have said that logarithms are in general incommensurable numbers. Their values can therefore only be given approximately.

If the value of any number is given to seven significant figures, then the **error** (i.e. the difference between the *given* value and the *exact* value of the number) is less than a millionth part of the number.

Example. 3·141592 is the value of π correct to seven significant figures. The *error* is less than ·000001; for π is less than 3·141593, and greater than 3·141592.

The ratio of ·000001 to 3·141592 is equal to 1 : 3141592. The ratio of ·000001 to π is less than this; i.e. much less than the ratio of one to one million.

217. An actual measurement of any kind must be made with the greatest care, with the most accurate instruments, by the most skilful observers, if it is to attain to anything like the accuracy represented by 'seven significant figures.'

Therefore the value of any quantity given correct to 'seven significant figures' is exact for all practical purposes.

218. We are given in the Tables the logarithms of all numbers from 1 to 99999; that is, of any number having *five* significant figures.

A Table consisting of the logarithms of all numbers from 1 to 9999999 (i.e. of any number having *seven* significant figures) would be *a hundred times as large.*

219. There is however a rule by which, if we are given a *complete* list of the logarithms of numbers having *five* significant figures, we can find the logarithms of numbers having *six* or *seven* significant figures.

Example. Suppose we require the logarithm of 4·804213.

From the Tables we find

$$\log 4{\cdot}8042 = {\cdot}6816211, \qquad \text{i.e. } 4{\cdot}8042 = 10^{\cdot 6816211\ldots},$$
$$\log 4{\cdot}8043 = {\cdot}6816301, \qquad 4{\cdot}8043 = 10^{\cdot 6816301\ldots}.$$

The number 4·804213 lies *between* the two numbers 4·8042, 4·8043 whose logarithms are found in the Tables, so that the required logarithm must lie *between* the two given logarithms.

Therefore we suppose that

$$\log 4{\cdot}804213 = {\cdot}6816211 + d, \qquad \text{i.e. } 4{\cdot}804213 = 10^{\cdot 6816211\ldots + d}.$$

220. The **RULE** is as follows. **The differences between three numbers are proportional to the corresponding differences between the logarithms of those numbers,** provided that the *differences* between the numbers are *small* compared with the numbers.

Example. Thus in the above example 4·8042, 4·8043 and 4·804213 are three numbers; ·6816211, ·6816301 and ·6816211 + d are their three logs.

The **difference** between the first and second numbers is ·0001.

The **difference** between the first and third numbers is ·000013.

The **difference** between the logarithms of the first and second numbers is ·000009.

The **difference** between the logarithms of the first and third numbers is d.

By the Rule these **differences** are in proportion

$$\therefore\ {\cdot}0001 : {\cdot}000013 = {\cdot}000009 : d,$$

or

$$100 : 13 = {\cdot}000009 : d;$$

whence

$$d = {\cdot}00000117\ldots,$$

$$\therefore\ \log 4{\cdot}804213 = {\cdot}6816211 + {\cdot}00000117\ldots$$
$$= {\cdot}68162227 = {\cdot}6816223 \text{ (to seven figures).}$$

221. We shall refer to the above rule as the **Rule of Proportional Differences.**

It is often called also 'The Principle of Proportional Parts.'

222. In Art. 197 we said that numbers are *not* proportional to their Logarithms. Hence the differences of numbers and the corresponding differences of their logarithms cannot be *exactly* in proportion. The rule is however true for all practical purposes. The proof of the rule belongs to a higher part of the subject than the present.

223. In the above example we said that

$$6{\cdot}68162227 = 6{\cdot}6816223;$$

and for this reason. We are *retaining* only *seven* significant figures in the decimal part of the logarithm.

If we put 6·6816222 for 6·68162227 the 'error' is greater than ·00000007.

If we put 6·6816223 for 6·68162227 the 'error' is less than ·00000003.

Thus the second error is less than the first.

In such a case, 1 must be added to the last digit which is retained, when the first digit which is neglected is 5 or greater than 5.

224. We give two more specimen examples.

Example 1. *Find the logarithm of* ·004804213.

We first find as before, by the rule of proportional differences, that

$$\log 4{\cdot}804213 = {\cdot}6816223$$

$$\therefore \log {\cdot}004804213 = \bar{3}{\cdot}6816223.$$

Example 2. *Find the number whose logarithm is* 2·5354291.

In the Table we find that

$$\cdot 5354207=\log 3\cdot 4310 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

and

$$\cdot 5354334=\log 3\cdot 4311 \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}.$$

Let

$$\cdot 5354291=\log (3\cdot 4310+d) \ldots\ldots\ldots\ldots\text{(iii)}.$$

Here we have three logarithms and three numbers.

The **difference** between the first and second logs is ·0000127.

The **difference** between the first and third logs is ·0000084.

The **difference** between the first and second numbers is ·0001.

The **difference** between the first and third numbers is d.

By the Rule these four **differences** are in proportion,

$$\therefore\ \cdot 0000127 : \cdot 0000084 = \cdot 0001 : d,$$

or,

$$127 : 84 = \cdot 0001 : d;$$

$$\therefore\ d = \cdot 0001 \times \tfrac{84}{127} = \cdot 0000661, \text{ etc.}$$

Therefore from (iii) $\cdot 5354291=\log (3\cdot 4310+\cdot 000066)$

$$=\log 3\cdot 431066.$$

Hence,

$$2\cdot 5354291=\log 343\cdot 1066,$$

or, the required number is 343·1066.

EXAMPLES. LVIII.

(1) Find log 7·65432, having given that

$$\log 7\cdot 6543 = \cdot 8839055,$$
$$\log 7\cdot 6544 = \cdot 8839112.$$

(2) Find log 564·123, having given that

$$\log 5\cdot 6412 = \cdot 7513715,$$
$$\log 5\cdot 6413 = \cdot 7513792.$$

(3) Find log ·0008736416, having given that

$$\log 8\cdot 7364 = \cdot 9413325,$$
$$\log 8\cdot 7365 = \cdot 9413375.$$

(4) Find log 6437125, having given that
log 6·4371 = ·8086903,
log 6·4372 = ·8086970.

(5) Find log 3·72456, having given that
log 37245 = 4·5710680,
log 37246 = 4·5710796.

(6) Find the number whose logarithm is ·5686760, having given that
·5686710 = log 3·7040,
·5686827 = log 3·7041.

(7) Find the number whose logarithm is 4·6602987, having given that
·6602962 = log 4·5740,
·6603057 = log 4·5741.

(8) Find the number whose logarithm is 6·3966938, having given that
·3966874 = log 2·4928,
·3967049 = log 2·4929.

(9) Find the number whose logarithm is $\bar{4}$·6431150, having given that
·6431071 = log 4·3965,
·6431170 = log 4·3966.

(10) Find the number whose logarithm is ·7550480, having given that
3·7550436 = log 5689·1,
2·7550512 = log 568·92.

225. The same Rule of Proportional Differences is used in the case of **angles** and their **Trigonometrical Ratios**; and therefore also in the case of **angles** and the **logarithms of their Ratios.**

Thus the (small) differences between three angles are assumed to be proportional to the corresponding differences between the sines of those three angles; also, proportional to the corresponding differences between the logarithms of the sines of those angles.

226. Sines and cosines are always less than unity, as also are the tangents of all angles between 0° and 45°.

The logarithms of these Ratios must therefore have *negative* characteristics.

To avoid the inconvenience of having to print these negative characteristics, the whole number 10 is added to each logarithm of the Trigonometrical Ratios, before it is set down in the Table.

The numbers thus recorded are called the **tabular logarithms** of the sine, cosine, etc., of an angle.

They are indicated by the letter 'L.'

Thus L *sin* 31° 15′, stands for the tabular logarithm of sin 31° 15′, and is equal to $\{\log(\sin 31^\circ 15') + 10\}$.

The words **logarithmic sine** are used as abbreviation for **tabular logarithm of the sine.**

Thus in the Tables we find

$$L \sin 31^\circ 15' = 9{\cdot}7149776.$$

Therefore $\log(\sin 31^\circ 15') = 9{\cdot}7149776 - 10 = \bar{1}{\cdot}7149776$.

Example 1. *Find* sin 31° 6′ 25″.

The Tables give $\sin 31^\circ 6' = {\cdot}5165333$ ....(i),

$\sin 31^\circ 7' = {\cdot}5167824$ ....(ii).

Let $\sin 31^\circ 7' 25'' = {\cdot}5165333 + d$ ....(iii).

The difference between the first two angles is 60″.

The difference between the first and third angle is 25″.

The differences between the corresponding sines are ·0002491 and *d*.

By the Rule these four **differences** are in proportion.

Therefore $60'' : 25'' = {\cdot}0002491 : d$,

$$\therefore d = {\cdot}0002491 \times \tfrac{25}{60} = {\cdot}0001038.$$

Hence from (iii) $\sin 31^\circ 7' 25'' = {\cdot}5165333 + {\cdot}0001038 = {\cdot}5166371$.

Example 2. Find the angle whose logarithmic cosine is 9·7858083.

The Table gives $9·7857611 = L\cos 52^\circ\, 22'$ ....(i),

$9·7859249 = L\cos 52^\circ\, 21'$ ....(ii).

The cosine diminishes as the angle increases. Hence corresponding to an increase in the angle there is a diminution of the cosine.

Hence, let $9·7858083 = L\cos(52^\circ\, 22' - D)$ ....(iii).

Subtracting the first tabular logarithm from the second the **difference** is ·0001638.

Subtracting the first tabular logarithm from the third, the **difference** is ·0000472.

Subtracting the first angle from the second, the **difference** is $-60''$.

Subtracting the first angle from the third, the **difference** is $-D$.

By the Rule these four **differences** are in proportion.

Therefore $\quad ·0001638 : ·0000472 = -60'' : -D,$

$$\therefore\ D = 60'' \times \tfrac{472}{1638} = 17·3''.$$

Hence $\quad 9·7858083 = L\cos(52^\circ\, 22' - 17'')$

$$= L\cos 52^\circ\, 21'\, 43''$$

EXAMPLES. LIX.

(1) Find $\sin 42^\circ\, 21'\, 30''$

having given that $\quad \sin 42^\circ\, 21' = ·6736577$

$\sin 42^\circ\, 22' = ·6738727.$

(2) Find $\cos 47^\circ\, 38'\, 30''$

having given that $\quad \cos 47^\circ\, 38' = ·6738727$

$\cos 47^\circ\, 39' = ·6736577.$

(3) Find $\cos 21^\circ\, 27'\, 45''$

having given that $\quad \cos 21^\circ\, 27' = ·9307370$

$\cos 21^\circ\, 28' = ·9306306.$

(4) Find the angle whose sine is ·6666666

having given that $\cdot6665325 = \sin 41^\circ\, 48'$

$\cdot6667493 = \sin 41^\circ\, 49'$.

(5) Find the angle whose cosine is ·3333333

having given that $\cdot3332584 = \cos 70^\circ\, 32'$

$\cdot3335326 = \cos 70^\circ\, 31'$.

(6) Find the angle whose cosine is ·25

having given that $\cdot2498167 = \cos 75^\circ\, 32'$

$\cdot2500984 = \cos 75^\circ\, 31'$.

(7) Find $L \sin 45^\circ\, 16'\, 30''$

having given that $L \sin 45^\circ\, 16' = 9\cdot8514969$

$L \sin 45^\circ\, 17' = 9\cdot8516220$.

(8) Find $L \tan 27^\circ\, 13'\, 45''$

having given that $L \tan 27^\circ\, 13' = 9\cdot7112148$

$L \tan 27^\circ\, 14' = 9\cdot7115254$.

(9) Find $L \cot 36^\circ\, 18'\, 20''$

having given that $L \cot 36^\circ\, 18' = 10\cdot1339650$

$L \cot 36^\circ\, 19' = 10\cdot1337003$.

(10) Find the angle whose Logarithmic tangent is 9·8464028,

having given that $9\cdot8463018 = L \tan 35^\circ\, 4'$

$9\cdot8465705 = L \tan 35^\circ\, 5'$.

(11) Find the angle whose Logarithmic cosine is 9·9448230,

having given that $9\cdot9447862 = L \cos 28^\circ\, 17'$

$9\cdot9448541 = L \cos 28^\circ\, 16'$.

(12) Find the angle whose Logarithmic cosecant is 10·4274623,

having given that $10\cdot4273638 = L \operatorname{cosec} 21^\circ\, 57'$

$10\cdot4276774 = L \operatorname{cosec} 21^\circ\, 56'$.

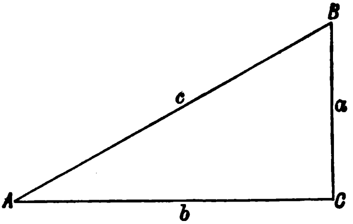
227. Problems in which each of the lines involved contains an *exact* number of feet, and each angle an *exact* number of degrees, **do not occur** in practical work.

As from time to time the skill of observers and of instrument-makers has increased, so also has the number of significant figures by which observations have been recorded.

Thus the want was felt of some method by which the labour involved in the multiplication and division of long numerical quantities could be avoided. At the end of the Seventeenth Century a celebrated Scotch mathematician, John Napier, Baron of Merchiston, proposed his method of 'Logarithms'; i.e. the method of representing numbers by **indices**; 'which, by reducing to a few days the labour of 'many months, doubles, as it were, the life of an Astronomer, 'besides freeing him from the errors and disgust inseparable 'from long calculations.' *Laplace.*

228. We shall now give a few examples of the **practical use** of logarithms.

Example 1. The sides containing the right angle C in a right angled triangle ABC contain 3456·4 ft. and 4543·5 feet respectively; find the angles of the triangle, and the length of the hypotenuse.



Let a, b, c be the lengths of the sides of the triangle opposite the angles A, B, C respectively.

Then $a=3456{\cdot}4$ feet, $b=4543{\cdot}5$ feet.

$$\tan A=\frac{a}{b}=\frac{3456{\cdot}4}{4543{\cdot}5}.$$

We shall give in two parallel columns the work (i) in the logarithm notation (ii) when expressed by indices.

In the Tables we find

$\log 3456{\cdot}4=3{\cdot}5386240.$	$3456{\cdot}4=10^{3\cdot5386240}.$
$\log 4543{\cdot}5=3{\cdot}6573905.$	$4543{\cdot}5=10^{3\cdot6573905}.$
$\therefore\ \log\frac{a}{b}=\log a-\text{ıog}\,b.$	$\therefore\ \frac{a}{b}=\frac{10^{3\cdot5386240}}{10^{3\cdot6573905}}.$
$=3{\cdot}5386240-3{\cdot}6573905.$	$=10^{3\cdot5386240-3\cdot6573905}.$
$\therefore\ \log\tan A=\bar{1}{\cdot}8812335.$	$\therefore\ \tan A=10^{\bar{1}\cdot8812335}.$
$\therefore\ L\tan A=9{\cdot}8812335.$	$=10^{9\cdot8812336-10}.$

In the Tables we find

$9{\cdot}8810522=L\tan 37^0\ 15'.$	$10^{9\cdot8810522-10}=\tan 37^0\ 15'.$
$9{\cdot}8813144=L\tan 37^0\ 16'.$	$10^{9\cdot8813144-10}=\tan 37^0\ 16'.$

Whence we find by the Rule of Proportional Differences

$9{\cdot}8812335=L\tan 37^0\ 15'\ 42''.$	$10^{9\cdot8812336-10}=\tan 37^0\ 15'\ 42''.$

$$\therefore\ A=37^0\ 15'\ 42''.$$

Also $B=(90^0-A)$, $\therefore\ B=52^0\ 44'\ 18''$,

and $\frac{c}{a}=\operatorname{cosec}A=\operatorname{cosec}37^0\ 15'\ 42''$,

$$\begin{aligned}\therefore\ \log c&=\log a+\log\operatorname{cosec}37^0\ 15'\ 42''\\&=\log a+L\operatorname{cosec}\ 37^0\ 15'\ 42''-10\\&=3{\cdot}5386240+10{\cdot}2179174\quad-10\\&=3{\cdot}7565414\\&=\log 5708{\cdot}8,\end{aligned}$$

$\therefore$ the hypotenuse contains 5708·8 feet.

Thus we have found the angles and the third side of the triangle.

229. There are some formulæ which are seldom used in practical work, because they are not adapted to logarithmic calculation. They are those in which powers of quantities are connected by the signs + or –.

Example. In the above example we might have found the length of the hypotenuse by means of the formula

$$c^2 = a^2 + b^2.$$

But we should have had to go through the process of calculating by multiplication the values of a^2 and b^2.

For this reason, a formula which consists entirely of **factors** is always preferred to one which consists of **terms**, when any of those terms contain any power of the quantities involved.

If in the above example, the lengths of the hypotenuse c and of one side a were given, then the formula

$$b^2 = c^2 - a^2 = (c - a)(c + a)$$

will give the length of b. For

$$\log b^2 = \log \{(c - a)(c + a)\},$$

or,

$$2 \log b = \log (c - a) + \log (c + a).$$

And the values of $(c + a)$ and $(c - a)$ are easily written down from the given values of c and a.

EXAMPLES. LX.

In the following questions A, B, C are the angles of a right angled triangle of which C is a right angle, and a, b, c are the lengths of the sides opposite those angles respectively.

(1) Given that $a = 1046{\cdot}7$ yards, $c = 1856{\cdot}2$ yards, $C = 90^\circ$, find A.

$\log 1046{\cdot}7 = 3{\cdot}0198222$, $\log 1856{\cdot}2 = 3{\cdot}2686248$,
$L \sin 34^\circ\, 19' = 9{\cdot}7510991$,
$L \sin 34^\circ\, 20' = 9{\cdot}7512842$.

(2) Given that $a=843{\cdot}2$ feet, $C=90^0$, and $A=34^0\,15'$; find c.

$\log 843{\cdot}2=2{\cdot}9259306$, $L \operatorname{cosec} 34^0\,15'=10{\cdot}2496421$,

$\log 1{\cdot}4982={\cdot}17557$.

(3) Given that $a=4845$ yards, $b=4742$ yards, and $C=90$, find A.

$\log 4845=3{\cdot}6852938$, $\log 4742=3{\cdot}6759615$,

$L \tan 45^0\,36'=10{\cdot}0090965$, $L \tan 45^0\,37'=10{\cdot}0093492$.

(4) Given that $c=8762$ feet, $C=90$, and $A=37^0\,10'$, find a and b.

$\log 8762=3{\cdot}9426032$, $L \sin 37^0\,10'=9{\cdot}7811344$,

$L \cos 37^0\,10'=9{\cdot}9013938$, $\log 5{\cdot}2934={\cdot}72373$,

$\log 6{\cdot}9823={\cdot}843997$.

(5) Given that $b=1694{\cdot}2$ chains, $C=90^0$, and $A=18^0\,47'$, find a.

$\log 1694{\cdot}2=3{\cdot}2289647$, $L \cot 18^0\,47'=10{\cdot}4683893$,

$\log 5{\cdot}7620={\cdot}76057$.

(6) Given that $a=1072$ chains, $c=4849$ chains, and $C=90^0$, find b.

$\log 5921=3{\cdot}7723951$, $\log 3777=3{\cdot}5771470$,

$\log 4{\cdot}729={\cdot}67477$.

(7) Given that $b=841$ feet, $c=3762$ feet, and $C=90^0$, find a.

$\log 4603=3{\cdot}6630410$, $\log 2921=3{\cdot}4655316$,

$\log 3{\cdot}6668={\cdot}56428$.

(8) Given that $a=7694{\cdot}5$ chains, $b=8471$ chains, $C=90^0$, find A and c.

$\log 7694{\cdot}5=3{\cdot}8861804$, $\log 8471=3{\cdot}9279347$,

$L \tan 42^0\,15'=9{\cdot}95824$, $L \operatorname{cosec} 42^0\,15'=10{\cdot}1723937$,

$\log 1{\cdot}1444={\cdot}05857$.

230. In the following examples the student must find the necessary logarithms from the Tables.

* MISCELLANEOUS EXAMPLES. LXI.

(1) A balloon is at a height of 2500 feet above a plain and its angle of elevation at a point in the plain is $40^\circ 35'$. How far is the balloon from the point of observation?

(2) A tower standing on a horizontal plain subtends an angle of $37^\circ 19' 30''$ at a point in the plain distant 369·5 feet from the foot of the tower. Find the height of the tower.

(3) The shadow of a tower on a horizontal plain in the sunlight is observed to be 176·23 feet and the elevation of the sun at that moment is $33^\circ 12'$. Find the height of the tower.

(4) From the top of a tower 163·5 feet high by the side of a river the angle of depression of a post on the opposite bank of the river is $29^\circ 47' 18''$. Find the distance of the post from the foot of the tower.

(5) Given $a=673{\cdot}12$, $b=415{\cdot}89$ chains, $C=90^\circ$, find A and B.

(6) Given $a=576{\cdot}12$, $c=873{\cdot}14$ chains, $C=90^\circ$, find b and A.

(7) From the top of a light-house 112·5 feet high, the angles of depression of two ships, when the line joining the ships points to the foot of the light-house, are $27^\circ 18'$ and $20^\circ 36'$ respectively. Find the distance between the ships.

(8) From the top of a cliff the angles of depression of the top and bottom of a light-house 97·25 feet high are observed to be $23^\circ 17'$ and $24^\circ 19'$ respectively. How much higher is the cliff than the light-house?

(9) Find the distance in space travelled in an hour, in consequence of the earth's rotation, by St Paul's cathedral. (Latitude of London$=51^\circ 25'$, earth's diameter$=7914$ miles.)

(10) The angle of elevation of a balloon from a station due south of it is $47^\circ 18' 30''$, and from another station due west of the former and distant 671·38 feet from it the elevation is $41^\circ 14'$. Find the height of the balloon.

CHAPTER XVI.

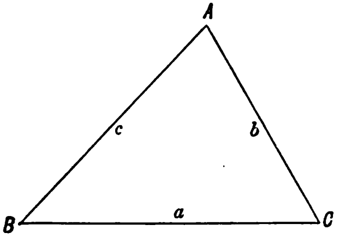
ON THE RELATIONS BETWEEN THE SIDES AND ANGLES OF A TRIANGLE.

231. The **three sides** and the **three angles** of any **triangle,** are called its six **parts.**

By the letters A, B, C we shall indicate

geometrically, the three angular points of the triangle ABC;

algebraically, the three angles at those angular points respectively.



By the letters a, b, c we shall indicate the measures of the sides BC, CA, AB opposite the angles A, B, C respectively.

232. I. We know that, $A + B + C = 180°$. [Euc. I. 32.]

233. Also if A be an angle of a triangle, then A may have any value between $0°$ and $180°$. Hence,

(i) **sin A** must be positive (and less than 1),

(ii) **cos A** may be positive or negative (but must be numerically less than 1),

(iii) **tan A** may have any value whatever, positive or negative.

234. Also, if we are given the value of

(i) **sin A**, there are **two** angles, each less than 180°, which have the given positive value for their sine.

(ii) **cos A**, or (iii) **tan A**, then there is only **one** value of A, which value can be found from the Tables.

235. $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=90^\circ$. Therefore $\frac{A}{2}$ is less than 90°, and its Trigonometrical Ratios are all positive. Also, $\frac{A}{2}$ is known, when the value of any one of its Ratios is given. Similar remarks of course apply to the angles B and C.

Example 1. *To prove* $\sin(A+B)=\sin C$.

$$A+B+C=180^0 \therefore A+B=180^0-C,$$

and ∴ $\sin(A+B)=\sin(180^0-C)=\sin C$. [p. 104.]

Example 2. *To prove* $\sin\frac{A+B}{2}=\cos\frac{C}{2}$,

Now $\frac{A+B+C}{2}=90^0. \quad \therefore \frac{A+B}{2}=90^0-\frac{C}{2}$,

and ∴ $\sin\frac{A+B}{2}=\sin\left(90^0-\frac{C}{2}\right)=\cos\frac{C}{2}$. [Art. 118.]

Example 3. *To prove*

$$\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$$

Now $\sin A+\sin B=2\sin\frac{A+B}{2}\cdot\cos\frac{A-B}{2}$. [Art. 157.]

$=2\cos\frac{C}{2}\cdot\cos\frac{A-B}{2}$. [Art. 118.]

and $\sin C=2\sin\frac{C}{2}\cdot\cos\frac{C}{2}$. [Art. 166.]

$=2\cos\frac{A+B}{2}\cdot\cos\frac{C}{2}$. [Art. 118.]

$$\therefore\ \sin A+\sin B+\sin C=2\cos\frac{C}{2}\,.\,\cos\frac{A-B}{2}+2\cos\frac{C}{2}\,.\,\cos\frac{A+B}{2},$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\right).$$

$$=2\cos\frac{C}{2}\left(2\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\right).\qquad\text{[Art. 157.]}$$

$$=4\cos\frac{A}{2}\,.\,\cos\frac{B}{2}\,.\,\cos\frac{C}{2}.\qquad\text{Q.E.D.}$$

EXAMPLES. LXII.

Find A from each of the six following equations, A being an angle of a triangle.

(1) $\cos A=\frac{1}{2}$. (2) $\cos A=-\frac{1}{2}$. (3) $\sin A=\frac{1}{2}$.

(4) $\tan A=-1$. (5) $\sin A=\dfrac{1}{\sqrt{2}}$ (6) $\tan A=-\sqrt{3}$.

Prove the following statements, A, B, C being the angles of a triangle.

(7) $\sin(A+B+C)=0$. (8) $\cos(A+B+C)=-1$.

(9) $\sin\dfrac{A+B+C}{2}=1$. (10) $\cos\dfrac{A+B+C}{2}=0$.

(11) $\tan(A+B)=-\tan C$. (12) $\cot\dfrac{B+C}{2}=\tan\dfrac{A}{2}$.

(13) $\cos(A+B)=-\cos C$. (14) $\cos(A+B-C)=-\cos 2C$.

(15) $\tan A-\cot B=\cos C\,.\,\sec A\,.\,\operatorname{cosec} B$.

(16) $\dfrac{\sin A-\sin B}{\sin A+\sin B}=\tan\dfrac{C}{2}\,.\,\tan\dfrac{A-B}{2}$.

(17) $\dfrac{\sin 3B-\sin 3C}{\cos 3C-\cos 3B}=\tan\dfrac{3A}{2}$.

(18) $\sin A+\sin B-\sin C=4\sin\dfrac{A}{2}\,.\,\sin\dfrac{B}{2}\,.\,\cos\dfrac{C}{2}$.

*(19) $\sin A-\sin B+\sin C=4\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}.$

(20) $\sin\frac{A}{2}\cdot\cos\frac{A}{2}+\sin\frac{B}{2}\cdot\cos\frac{B}{2}+\sin\frac{C}{2}\cdot\cos\frac{C}{2}$
$=2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}.$

(21) $\cos A+\cos B+\cos C-1=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$

(22) $\cos^2\frac{A}{2}+\cos^2\frac{B}{2}-\cos^2\frac{C}{2}=2\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\sin\frac{C}{2}.$

(23) $\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\cos\frac{C}{2}.$

(24) $\sin\frac{B+C-A}{2}+\sin\frac{C+A-B}{2}+\sin\frac{A+B-C}{2}-1$
$=4\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}.$

(25) $\sin 2A+\sin 2B+\sin 2C=4\sin A\cdot\sin B\cdot\sin C.$

(26) $\sin A\cdot\cos A-\sin B\cdot\cos B+\sin C\cdot\cos C$
$=2\cos A\cdot\sin B\cdot\cos C.$

(27) $\sin(B+C-A)-\sin(C+A-B)+\sin(A+B-C)$
$=4\cos A\cdot\sin B\cdot\cos C.$

(28) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cdot\cos B\cdot\cos C.$

(29) $\sin^2 A-\sin^2 B+\sin^2 C=2\sin A\cdot\cos B\cdot\sin C.$

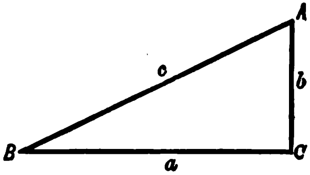
(30) $\cos(B+C-A)+\cos(C+A-B)-\cos(A+B-C)+1$
$=4\sin A\cdot\sin B\cdot\cos C.$

(31) $\sin\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}+\sin\frac{B}{2}\cdot\cos\frac{C}{2}\cdot\cos\frac{A}{2}$
$+\sin\frac{C}{2}\cdot\cos\frac{A}{2}\cdot\cos\frac{B}{2}=\sin\frac{A}{2}\cdot\sin\frac{B}{2}\cdot\sin\frac{C}{2}+1.$

(32) $\tan A+\tan B+\tan C=\tan A\cdot\tan B\cdot\tan C.$

(33) $\tan\frac{B}{2}\cdot\tan\frac{C}{2}+\tan\frac{C}{2}\cdot\tan\frac{A}{2}+\tan\frac{A}{2}\cdot\tan\frac{B}{2}=1.$

236. If ABC be a **right-angled** triangle having $C = 90°$, then $(A + B) = 90°$.



Hence, $\sin A = \sin (90° - B) = \cos B$.

Also, $\sin A = \dfrac{a}{c} = \cos B$,

and so on. (See page 52.)

EXAMPLES. LXIII.

In a right-angled triangle ABC, in which C is a right angle, prove the following statements.

(1) $\tan A = \cot B$.

(2) $\tan B = \cot A + \cos C$.

(3) $\sin 2A = \sin 2B$.

(4) $\cos 2A + \cos 2B = 0$.

(5) $\sin 2A = \dfrac{2ab}{c^2}$.

(6) $\operatorname{cosec} 2B = \dfrac{a}{2b} + \dfrac{b}{2a}$.

(7) $\cos 2A = \dfrac{b^2 - a^2}{c^2}$.

(8) $\cos 2B = \dfrac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$.

(9) $\sin^2 \dfrac{B}{2} = \dfrac{c - a}{2c}$.

(10) $\cos^2 \dfrac{A}{2} = \dfrac{c + b}{2c}$.

(11) $\left(\cos \dfrac{A}{2} + \sin \dfrac{A}{2}\right)^2 = \dfrac{a + c}{c}$.

(12) $\dfrac{a - b}{a + b} = \tan \dfrac{A - B}{2}$.

(13) $\sin (A - B) + \cos 2A = 0$.

(14) $\sin (A - B) + \sin (2A + C) = 0$.

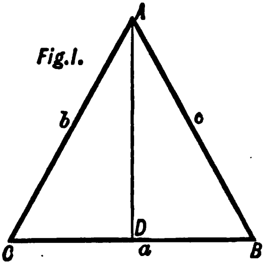
(15) $(\sin A - \sin B)^2 + (\cos A + \cos B)^2 = 2$.

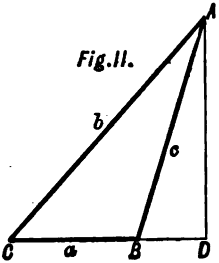
(16) $\sqrt{\dfrac{a + b}{a - b}} + \sqrt{\dfrac{a - b}{a + b}} = \dfrac{2 \sin A}{\sqrt{\cos 2B}}$.

237. II. *To prove* $a = b\cos C + c\cos B$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary.

There will be three cases. Fig. i. when both B and C are acute angles; Fig. ii. when one of them (B) is obtuse; Fig. iii. when one of them (B) is a right angle. Then,





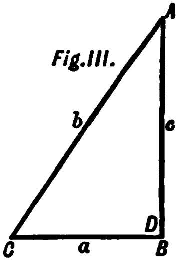


Fig. i. $$\frac{CD}{CA} = \cos ACD\,; \text{ or, } CD = b\cos C,$$

$$\text{and } \frac{DB}{AB} = \cos ABD\,; \text{ or, } DB = c\cos B,$$

$$\therefore a = CD + DB = b\cos C + c\cos B.$$

Fig. ii. $$\frac{CD}{CA} = \cos ACD\,; \text{ or, } CD = b\cos C,$$

$$\frac{BD}{AB} = \cos ABD\,; \text{ or, } BD = c\cos(180^\circ - B),$$

$$\therefore a = CD - BD = b\cos C - c\cos(180^\circ - B)$$
$$= b\cos C + c\cos B.$$

Fig. iii. $a = CB = b\cos C$

$$= b\cos C + c\cos B. \quad [\text{For, } \cos B = \cos 90^\circ = 0.]$$

Similarly it may be proved that,

$$b = c\cos A + a\cos C\,; \quad c = a\cos B + b\cos A.$$

238. III. *To prove that in any triangle, the sides are proportional to the sines of the opposite angles;* or, *To prove that* $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

From A, any one of the angular points, draw AD perpendicular to BC, or to BC produced if necessary. Then,

Fig. i. $AD = b \sin C$; for, $\frac{AD}{AC} = \sin C$ [Def.]:

$$\text{also } AD = c \sin B; \text{ for, } \frac{AD}{AB} = \sin B.$$

$$\therefore b \sin C = c \sin B;$$

$$\text{or, } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Fig. ii. $AD = b \sin C$,

$$\text{and } AD = c \sin ABD = c \sin (180^\circ - B)$$

$$\therefore AD = c \sin B;$$

$$\therefore b \sin C = c \sin B;$$

$$\text{or, } \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Fig. iii. $AB = AC \,.\, \sin C$; or, $c = b \sin C$;

$$\therefore \frac{c}{\sin C} = \frac{b}{\sin B}. \quad [\text{For } \sin B = \sin 90^\circ = 1.]$$

Similarly it may be proved that

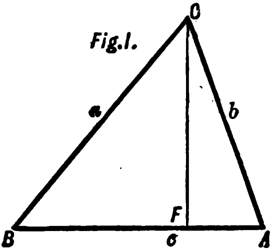
$$\frac{a}{\sin A} = \frac{b}{\sin B};$$

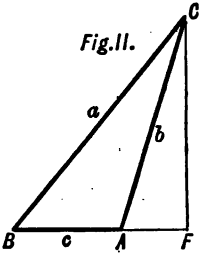
$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \quad \text{Q.E.D.}$$

239. IV. *To prove that* $a^2 = b^2 + c^2 - 2bc \cos A$.

Take one of the angles A. Then of the other two, one must be acute. Let B be an acute angle. From C draw CF perpendicular to BA, or to BA produced if necessary.

There will be three figures according as A is less, greater than, or equal to a right angle. Then,





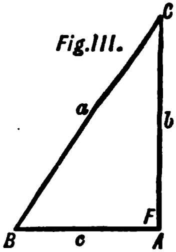


Fig. i. $BC^2 = CA^2 + AB^2 - 2 . BA . FA$; [Euc. II. 13]

or, $a^2 = b^2 + c^2 - 2c . FA$

$= b^2 + c^2 - 2cb \cos A$. [For $FA = b . \cos A$.]

Fig. ii. $BC^2 = CA^2 + AB^2 + 2 . BA . AF$; [Euc. II. 12]

or, $a^2 = b^2 + c^2 + 2cb \cos FAC$

$= b^2 + c^2 - 2bc \cos A$. [For $FAC = 180° - A$.]

Fig. iii. $BC^2 = CA^2 + AB^2$; [Euc. I. 47]

or, $a^2 = b^2 + c^2 - 2bc \cos A$. [For $\cos A = \cos 90° = 0$.]

Similarly it may be proved that

$$b^2 = c^2 + a^2 - 2ca \cos B,$$

and that

$$c^2 = a^2 + b^2 - 2ab \cos C.$$

240. V. Hence,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \quad \cos B = \frac{c^2 + a^2 - b^2}{2ca}, \quad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

241. VI. *To prove that* $\sin^2 \frac{A}{2} = \frac{(a+c-b)(a+b-c)}{4bc}$,

and that $$\cos^2 \frac{A}{2} = \frac{(a+b+c)(b+c-a)}{4bc}.$$

Since $\cos A = 1 - 2\sin^2 \frac{A}{2}$, [Art. 166]

and $\cos A = \frac{b^2+c^2-a^2}{2bc}$; [Art. 240]

$$\therefore\ 2\sin^2 \frac{A}{2} = 1 - \cos A = 1 - \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{2bc-(b^2+c^2-a^2)}{2bc} = \frac{a^2-(b^2-2bc+c^2)}{2bc}$$

$$= \frac{a^2-(b-c)^2}{2bc} = \frac{\{a-(b-c)\}\{a+(b-c)\}}{2bc}$$

$$\therefore\ \sin^2 \frac{A}{2} = \frac{(a+c-b)(a+b-c)}{4bc}. \quad \text{Q.E.D.}$$

Again, $\cos A = 2\cos^2 \frac{A}{2} - 1$; [Art. 166]

$$\therefore\ 2\cos^2 \frac{A}{2} = 1 + \cos A = 1 + \frac{b^2+c^2-a^2}{2bc}$$

$$\therefore\ \cos^2 \frac{A}{2} = \frac{(b+c)^2-a^2}{4bc} = \frac{(b+c+a)(b+c-a)}{4bc}. \quad \text{Q.E.D.}$$

242. Similarly it may be proved that

$$\sin^2 \frac{B}{2} = \frac{(b+c-a)(b+a-c)}{4ca},\quad \cos^2 \frac{B}{2} = \frac{(a+b+c)(a+c-b)}{4ca},$$

and that

$$\sin^2 \frac{C}{2} = \frac{(c+a-b)(c+b-a)}{4ab},\quad \cos^2 \frac{C}{2} = \frac{(a+b+c)(a+b-c)}{4ab}.$$

243. VII. Now let s stand for $\dfrac{a+b+c}{2}$, so that $(a+b+c)=2s$.

Then, $(b+c-a)=(b+c+a-2a)=(2s-2a)=2(s-a)$,

and $(c+a-b)=(c+a+b-2b)=(2s-2b)=2(s-b)$,

and $(a+b-c)=(a+b+c-2c)=(2s-2c)=2(s-c)$.

Then the result of Art. 241 may be written

$$\sin^2\frac{A}{2}=\frac{2(s-b)\,2(s-c)}{4bc}\,;\quad\text{or, } \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\text{and } \cos^2\frac{A}{2}=\frac{2s\,2(s-a)}{4bc}\,;\quad\text{or, } \cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

and so on.

Hence, $$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)}}.$$

244. VIII. Again

$$\sin A=2\sin\frac{A}{2}\,.\cos\frac{A}{2}\,; \qquad \text{[Art. 166]}$$

$$\therefore \sin A=2\sqrt{\frac{(s-b)(s-c)}{bc}}\,.\sqrt{\frac{s(s-a)}{bc}}$$

$$=\frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

The letter S usually stands for $\sqrt{s(s-a)(s-b)(s-c)}$, so that the above may be written $\dfrac{\sin A}{a}=\dfrac{2S}{abc}$.

Similarly, $$\frac{\sin B}{b}=\frac{2S}{abc}=\frac{\sin C}{c}.$$

245. IX. *To prove that* $\frac{b-c}{b+c}\cdot\cot\frac{A}{2}=\tan\frac{B-C}{2}$.

Since $\frac{b}{\sin B}=\frac{c}{\sin C}$, let each of these fractions $=d$.

Then $\quad b=d\sin B$, and $c=d\sin C$.

$$\therefore \frac{b-c}{b+c}=\frac{d\sin B-d\sin C}{d\sin B+d\sin C}=\frac{\sin B-\sin C}{\sin B+\sin C}$$

$$=\frac{2\sin\frac{B-C}{2}\cdot\cos\frac{B+C}{2}}{2\sin\frac{B+C}{2}\cdot\cos\frac{B-C}{2}}=\frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}$$

$$=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}. \quad \left[\text{Since } \tan\frac{B+C}{2}=\tan\left(90^\circ-\frac{A}{2}\right).\right]$$

$$\therefore \frac{b-c}{b+c}\cdot\cot\frac{A}{2}=\frac{\tan\frac{B-C}{2}}{\cot\frac{A}{2}}\cdot\cot\frac{A}{2}=\tan\frac{B-C}{2}. \qquad \text{Q.E.D.}$$

Similarly, $\quad \frac{c-a}{c+a}\cdot\cot\frac{B}{2}=\tan\frac{C-A}{2}$,

and $\quad \frac{a-b}{a+b}\cdot\cot\frac{C}{2}=\tan\frac{A-B}{2}$.

246. NOTE. In the next chapter we shall prove, that given three of the parts of a triangle (one of which must be a side), we can find the remaining three parts by means of the formulæ of this chapter. Hence, these formulæ cannot be equivalent to more than three *independent* equations. It is an instructive, but somewhat difficult problem to take three of these formulæ (*e.g.* $A+B+C=180^\circ$, $b=c\cos A+a\cos C$, $c=b\cos A+a\cos B$) and deduce all the others from them.

247. The following examples are important.

Example 1. Suppose we are given that

$$\left.\begin{aligned} a&=b\cos C+c\cos B,\\ b&=c\cos A+a\cos C,\\ c&=a\cos B+b\cos A.\end{aligned}\right\}$$

Then, taking a times the first $+b$ times the second $-c$ times the third, we get

$$a^2+b^2-c^2=(ab\cos C+ac\cos B)+(bc\cos A+ba\cos C)$$
$$-(ca\cos B+cb\cos A),$$
$$=2ab\cos C.$$

Example 2. Suppose we are given that $A+B+C=180^\circ$,

and that $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}\,[=d],$$

then $$\frac{b^2+c^2-a^2}{2bc}=\frac{d^2\sin^2 B+d^2\sin^2 C-d^2\sin^2 A}{2d^2\sin B\sin C},$$

$$=\frac{\sin^2 B+\sin^2 C-\sin^2 A}{2\sin B\sin C}=\frac{2\cos A\,.\,\sin B\,.\,\sin C}{2\sin B\,.\,\sin C}$$

[Ex. 29, p. 194, since $(A+B+C)=180^\circ$.]

$$=\cos A.$$

248. The formula

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=d$$

is very frequently of use in solving examples.

Example. Prove that

$$a\cos A+b\cos B+c\cos C=2a\sin B\sin C.$$

Since $a=d\sin A,\ b=d\sin B,\ c=d\sin C,$

the above may be written

$$d\sin A\,.\,\cos A+d\sin B\,.\,\cos B+d\sin C\,.\,\cos C$$
$$=2d\sin A\,.\,\sin B\,.\,\sin C,$$

or $$\sin 2A+\sin 2B+\sin 2C=4\sin A\,.\,\sin B\,.\,\sin C,$$

which is Example (25) on page 194.

249. The student is advised to make himself thoroughly familiar with the following formulæ:

$$a = b\cos C + c\cos B \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}\,[=d] = \frac{abc}{2S} \;\ldots\ldots\ldots\ldots \text{(iii)},$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(v)}.$$

$$\left.\begin{aligned}\sin\frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}\\ \cos\frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots \text{(vii)},$$

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)} = \frac{2S}{bc}\ldots\ldots \text{(viii)},$$

$$\tan\frac{B-C}{2} = \frac{b-c}{b+c}\,.\cot\frac{A}{2} \;\ldots\ldots\ldots\ldots\ldots\ldots \text{(ix)}.$$

EXAMPLES. LXIV.

In any triangle ABC prove the following statements:

(1) $\dfrac{\sin A + 2\sin B}{a+2b} = \dfrac{\sin C}{c}$.

(2) $\dfrac{\sin^2 A - m.\sin^2 B}{a^2 - m.b^2} = \dfrac{\sin^2 C}{c^2}$.

(3) $a\cos A + b\cos B - c\cos C = 2c\cos A.\cos B$.

(4) $(a+b)\sin\dfrac{C}{2} = c\cos\dfrac{A-B}{2}$.

(5) $(b-c)\cos\dfrac{A}{2} = a\sin\dfrac{B-C}{2}$.

(6) $\dfrac{\cos A}{\sin B.\sin C} + \dfrac{\cos B}{\sin C.\sin A} + \dfrac{\cos C}{\sin A.\sin B} = 2$.

(7) $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

(8) $\dfrac{a-b}{c}=\dfrac{\cos B-\cos A}{1+\cos C}.$

(9) $\dfrac{b+c}{a}=\dfrac{\cos B+\cos C}{1-\cos A}.$

(10) $\sqrt{bc\sin B\,.\,\sin C}=\dfrac{b^2\sin C+c^2\sin B}{b+c}.$

(11) $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C.$

(12) $b+c-a=(b+c)\cos A-(c-a)\cos B+(a-b)\cos C.$

(13) $\tan A=\dfrac{a\sin C}{b-a\cos C}.$

(14) $a(b^2+c^2)\cos A+b(c^2+a^2)\cos B+c(a^2+b^2)\cos C=3abc.$

(15) $a\cos(A+B+C)-b\cos(B+A)-c\cos(A+C)=0.$

(16) $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}.$

(17) $\dfrac{\tan B}{\tan C}=\dfrac{a^2+b^2-c^2}{a^2-b^2+c^2}.$ (18) $b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}=s.$

(19) $\tan\dfrac{B}{2}\,.\,\tan\dfrac{C}{2}=\dfrac{b+c-a}{b+c+a}.$

(20) $\tan\dfrac{A}{2}(b+c-a)=\tan\dfrac{B}{2}(c+a-b).$

(21) $c^2=(a+b)^2\sin^2\dfrac{C}{2}+(a-b)^2\cos^2\dfrac{C}{2}.$

* * MISCELLANEOUS EXAMPLES. LXV.

(1) If p is the length of the perpendicular from A on BC,

$$\sin A=\frac{ap}{bc}.$$

(2) If $2\cos B\,.\,\sin C=\sin A$, then $B=C$.

(3) If $A=3B$, then $\sin B=\frac{1}{2}\sqrt{\dfrac{3b-a}{b}}.$

(4) If $\sqrt{bc \sin B \,.\, \sin C} = \dfrac{b^2 \sin B + c^2 \sin C}{b+c}$, then $B=C$.

(5) $a \cos \dfrac{B}{2} \,.\, \cos \dfrac{C}{2} \,.\, \operatorname{cosec} \dfrac{A}{2} = b \cos \dfrac{C}{2} \,.\, \cos \dfrac{A}{2} \,.\, \operatorname{cosec} \dfrac{B}{2}$

$$= c \cos \frac{A}{2} \,.\, \cos \frac{B}{2} \,.\, \operatorname{cosec} \frac{C}{2} = s.$$

(6) Given $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$, prove that $\cos C = -\frac{16}{65}$.

(7) If $\sin^2 B + \sin^2 C = \sin^2 A$, then $A = 90^0$.

(8) If $\sin 2B + \sin 2C = \sin 2A$, then either $B = 90^0$ or $C = 90^0$.

(9) If $A : B : C = 1 : 2 : 5$, then $1 + 4 \cos A \,.\, \cos B \,.\, \cos C = 0$, and a^2, b^2, c^2 are in A.P.

(10) $a \sin \dfrac{A}{2} \,.\, \sin \dfrac{B-C}{2} + b \sin \dfrac{B}{2} \,.\, \sin \dfrac{C-A}{2}$

$$+ c \sin \frac{C}{2} \,.\, \sin \frac{A-B}{2} = 0.$$

(11) If D is the middle point of BC, prove that

$$4AD^2 = 2b^2 + 2c^2 - a^2.$$

(12) Given that $a = 2b$, and that $A = 3B$, prove that $C = 60^0$.

(13) $abc\,(a \cos A + b \cos B + c \cos C) = 8S^2$.

(14) If $b \cos^2 \dfrac{C}{2} + c \cos^2 \dfrac{B}{2} = \dfrac{3a}{2}$, then b, a, c are in A.P.

(15) If D, E, F are the middle points of the sides BC, CA, AB, then

$$4\,(AD^2 + BE^2 + CF^2) = 3\,(a^2 + b^2 + c^2).$$

(16) If D is the middle point of BC, $\cot ADB = \dfrac{b^2 - c^2}{4S}$.

(17) If d, e, f are the perpendiculars from A, B, C on the opposite sides of the triangle, then

$$a \sin A + b \sin B + c \sin C = 2\,(d \cos A + e \cos B + f \cos C).$$

[Some of the Examples in the Appendix might be worked by the student at this stage.]

CHAPTER XVII.

ON THE SOLUTION OF TRIANGLES.

250. The problem known as the **Solution of Triangles** may be stated thus: *When a sufficient number of the parts of a triangle are given, to find the magnitude of each of the other parts.*

251. When **three** parts of a Triangle (one of which must be a side) are given, the other parts can in general be determined.

There are four cases.

I. Given **three sides.** [Compare Euc. I. 8.]

II. Given **one side** and **two angles.** [Euc. I. 26.]

III. Given **two sides** and the **angle between** them. [Euc. I. 4.]

IV. Given **two sides** and the **angle opposite** one of them. [Compare Euc. VI. 7.]

Case I.

252. Given **three sides,** a, b, c. [Euc. I. 8; VI. 5.]

We find two of the angles from the formulæ

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}.$$

The third angle $C = 180 - A - B$.

253. In practical work we proceed as follows:

$$\log\tan\frac{A}{2}=\log\sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

or,

$$L\tan\frac{A}{2}-10=\tfrac{1}{2}\{\log(s-b)+\log(s-c)-\log s-\log(s-a)\}.$$

Similarly,

$$L\tan\frac{B}{2}-10=\tfrac{1}{2}\{\log(s-c)+\log(s-a)-\log s-\log(s-b)\}.$$

254. Either of the formulæ $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$, $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ may also be used as above.

The $\sin\frac{A}{2}$ and the $\cos\frac{A}{2}$ formulæ are either of them as convenient as the $\tan\frac{A}{2}$ formula, when *one* of the angles only is to be found. If *all* the angles are to be found the tangent formula is convenient, because we can find the L tangents of two half angles from the same *four* logs, viz. $\log s$, $\log(s-a)$, $\log(s-b)$, $\log(s-c)$. To find the L sines of *two* half angles we require the *six* logarithms, viz. $\log(s-a)$, $\log(s-b)$, $\log(s-c)$, $\log a$, $\log b$, $\log c$.

Example. Given $a=275{\cdot}35$, $b=189{\cdot}28$, $c=301{\cdot}47$ chains, find A and B.

Here, $s=383{\cdot}05$, $s-a=107{\cdot}70$, $s-b=193{\cdot}77$, $s-c=81{\cdot}58$.

Then

$$L\tan\frac{A}{2}=10+\tfrac{1}{2}\{\log 193{\cdot}77+\log 81{\cdot}58-\log 383{\cdot}05-\log 107{\cdot}70\}$$
$$=10+\tfrac{1}{2}\{2{\cdot}2872865+1{\cdot}9115837-2{\cdot}5832555-2{\cdot}0322157\}$$
$$=9{\cdot}7916995 \qquad \text{[from the tables]},$$

whence $\frac{A}{2}=31^\circ\,45'\,28{\cdot}5''$; $\therefore A=63^\circ\,30'\,57''$.

Also

$$L\tan\frac{B}{2}=10+\tfrac{1}{2}\{\log 81{\cdot}58+\log 107{\cdot}70-\log 383{\cdot}05-\log 193{\cdot}77\}$$

$$=9{\cdot}5366287=L\tan 18^\circ\,59'\,9{\cdot}8'';$$

$$\therefore\ B=37^\circ\,58'\,20'';\quad C=180^\circ-A-B=78^\circ\,30'\,43''.$$

255. This Case may also be solved by the formula

$$\cos A=\frac{b^2+c^2-a^2}{2bc}.$$

But this formula is not adapted for logarithmic calculation, and therefore is seldom used in practice.

It may sometimes be used with advantage, when the given lengths of a, b, c each contain less than three digits.

Example. Find the greatest angle of the triangle whose sides are 13, 14, 15.

Let $a=15$, $b=14$, $c=13$. Then the greatest angle is A.

$$\text{Now, }\cos A=\frac{14^2+13^2-15^2}{2\times14\times13}=\frac{140}{2\times14\times13}=\tfrac{5}{13}=\cdot384615.$$

$$=\cos 67^\circ\,23',\text{ nearly.}$$

[By the Table of natural cosines.]

$$\therefore\ \text{the greatest angle}=67^\circ\,23'.$$

EXAMPLES. LXVI.

(1) If $a=352{\cdot}25$, $b=513{\cdot}27$, $c=482{\cdot}68$ yards, find the angle A, having given

$$\log 674{\cdot}10=2{\cdot}8287243,\quad \log 321{\cdot}85=2{\cdot}5076535,$$

$$\log 160{\cdot}83=2{\cdot}2063401,\quad \log 191{\cdot}42=2{\cdot}2819873,$$

$$L\tan 20^\circ\,38'=9{\cdot}5758104,\quad L\tan 20^\circ\,39'=9{\cdot}5761934.$$

(2) Find the two largest angles of the triangle whose sides are 484, 376, 522 chains, having given that

$$\log 6{\cdot}91 = {\cdot}8394780, \quad \log 3{\cdot}15 = {\cdot}4983106,$$
$$\log 2{\cdot}07 = {\cdot}3159703, \quad \log 1{\cdot}69 = {\cdot}2278867,$$
$$L \tan 36^\circ\, 46'\, 6'' = 9{\cdot}8734581, \quad L \tan 31^\circ\, 23'\, 9'' = 9{\cdot}7853745.$$

(3) If $a = 5238$, $b = 5662$, $c = 9384$ yards, find the angles A and B, having given

$$\log 1{\cdot}0142 = {\cdot}0061236, \quad \log 4{\cdot}904 = {\cdot}6905505,$$
$$\log 4{\cdot}48 = {\cdot}6512780, \quad \log 7{\cdot}58 = {\cdot}8796692,$$
$$L \tan 14^\circ\, 38' = 9{\cdot}4168099, \quad L \tan 15^\circ\, 57' = 9{\cdot}4560641,$$
$$L \tan 14^\circ\, 39' = 9{\cdot}4173265, \quad L \tan 15^\circ\, 58' = 9{\cdot}4565420.$$

(4) If $a = 4090$, $b = 3850$, $c = 3811$ yards, find A, having given

$$\log 5{\cdot}8755 = {\cdot}7690448, \quad \log 3{\cdot}85 = {\cdot}5854607,$$
$$\log 1{\cdot}7855 = {\cdot}2517599, \quad \log 3{\cdot}811 = {\cdot}5810389,$$
$$L \cos 32^\circ\, 15' = 9{\cdot}9272306, \quad L \cos 32^\circ\, 16' = 9{\cdot}9271509.$$

(5) Find the greatest angle in a triangle whose sides are 7 feet, 8 feet, and 9 feet, having given

$$\log 3 = {\cdot}4771213, \quad L \cos 36^\circ\, 42' = 9{\cdot}9040529,$$
$$\log 1{\cdot}4 = {\cdot}146128, \quad \text{diff. for } 60'' = {\cdot}0000942.$$

(6) Find the smallest angle of the triangle whose sides are 8 feet, 10 feet, and 12 feet, having given that

$$\log 2 = {\cdot}30103, \quad L \sin 20^\circ\, 42' = 9{\cdot}5483585, \quad \text{diff. for } 60'' = {\cdot}0003342.$$

(7) If $a : b : c = 4 : 5 : 6$, find C, having given

$$\log 2 = {\cdot}3010300, \quad \log 3 = {\cdot}4771213,$$
$$L \cos 41^\circ\, 25' = 9{\cdot}8750142, \quad \text{diff. for } 60'' = {\cdot}0001115.$$

(8) The sides of a triangle are 2, $\sqrt{6}$, and $1 + \sqrt{3}$, find the angles.

(9) The sides of a triangle are 2, $\sqrt{2}$ and $\sqrt{3} - 1$, find the angles.

Case II.

256. Given **one side** and **two angles**, as a, B, C.

[Euc. I. 26; VI. 4.]

First, $A = 180° - B - C$; which determines A.

Next, $$\frac{b}{\sin B} = \frac{a}{\sin A}, \quad \text{or, } b = \frac{a \,.\, \sin B}{\sin A};$$

and, $$\frac{c}{\sin C} = \frac{a}{\sin A}, \quad \text{or, } c = \frac{a \,.\, \sin C}{\sin A}.$$

These determine b and c.

257. In practical work we proceed as follows;

Since $b = \dfrac{a \,.\, \sin B}{\sin A}$,

$$\therefore \log b = \log \frac{a \,.\, \sin B}{\sin A}$$

$$\therefore \log b = \log a + \log (\sin B) + 10 - (10 + \log \sin A).$$

or, $$\log b = \log a + L \sin B - L \sin A.$$

Similarly, $\log c = \log a + L \sin C - L \sin A$.

Example. Given that $c = 1764{\cdot}3$ feet, $C = 18° 27'$, and $B = 66° 39'$, find b.

From the Tables we find $\log 1764{\cdot}3 = 3{\cdot}2465724$.

$$L \sin 18° 27' = 9{\cdot}5003421, \quad L \sin 66° 39' = 9{\cdot}9628904;$$

$$\therefore \log b = 3{\cdot}2465724 + 9{\cdot}9628904 - 9{\cdot}5003421$$

$$= 3{\cdot}7091207 = \log 5118{\cdot}2;$$

$$\therefore b = 5118{\cdot}2 \text{ feet.}$$

EXAMPLES. LXVII.

(1) If $A = 53^\circ\, 24'$, $B = 66^\circ\, 27'$, $c = 338{\cdot}65$ yards, find C and a, having given that

$$L \sin 53^\circ\, 24' = 9{\cdot}9046168, \quad \log 3{\cdot}3865 = {\cdot}5297511,$$
$$L \sin 60^\circ\;\; 9' = 9{\cdot}9381851, \quad \log 3{\cdot}1346 = {\cdot}4961821,$$
$$\log 3{\cdot}1347 = {\cdot}4961960.$$

(2) If $A = 48^\circ$, $B = 54^\circ$, and $c = 38$ inches, find a and b, having given that

$$\log 38 = 1{\cdot}5797836, \quad \log 2{\cdot}88704 = {\cdot}4604527,$$
$$\log 3{\cdot}14295 = {\cdot}4973368, \quad L \sin 54^\circ = 9{\cdot}9079576,$$
$$L \sin 78^\circ = 9{\cdot}9904044, \quad L \sin 48^\circ = 9{\cdot}8710735.$$

(3) Find c, having given that $a = 1000$ yards, $A = 50^\circ$, $C = 66^\circ$, and that

$$L \sin 50^\circ = 9{\cdot}8842540, \quad L \sin 66^\circ = 9{\cdot}9607302,$$
$$\log 1{\cdot}19255 = {\cdot}0764762.$$

(4) Find b, having given that $B = 32^\circ\, 15'$, $C = 21^\circ\, 47'\, 20''$, $a = 34$ feet.

$$\log 3{\cdot}4 = {\cdot}531479, \quad L \sin 32^\circ\, 15' = 9{\cdot}727228,$$
$$\log 2{\cdot}241 = {\cdot}350442, \quad L \sin 54^\circ\;\; 2' = 9{\cdot}908141,$$
$$\log 2{\cdot}242 = {\cdot}350636, \quad L \sin 54^\circ\;\; 3' = 9\;908233.$$

(5) Find a, b, C, having given $A = 72^\circ\, 4'$, $B = 41^\circ\, 56'\, 18''$, $c = 24$ feet.

$$\log 2{\cdot}4 = {\cdot}3802112, \quad L \sin 72^\circ\, 4' = 9{\cdot}9783702,$$
$$\log 1{\cdot}755 = {\cdot}2442771, \quad L \sin 41^\circ\, 56'\, 10'' = 9{\cdot}8249725,$$
$$\log 1{\cdot}756 = {\cdot}2445245, \quad L \sin 41^\circ\, 56'\, 20'' = 9{\cdot}8249959,$$
$$\log 2{\cdot}4995 = {\cdot}3978531, \quad L \sin 65^\circ\, 59' = 9{\cdot}9606739,$$
$$\log 2{\cdot}4996 = {\cdot}3978705, \quad L \sin 66^\circ = 9{\cdot}9607302.$$

Case III.

258. Given **two sides** and the **included angle**, as b, c, A. [Euc. I. 4; VI. 6.]

First, $B + C = 180° - A$. Thus $(B + C)$ is determined.

Next,
$$\tan\frac{B - C}{2} = \frac{b - c}{b + c}\cot\frac{A}{2}.$$

Thus $(B - C)$ is determined.

And B and C can be found when the values of $(B + C)$ and $(B - C)$ are known.

Lastly,
$$\frac{a}{\sin A} = \frac{b}{\sin B}, \text{ or } a = \frac{b \,.\, \sin A}{\sin B}.$$

Whence a is determined.

259. In practical work we proceed as follows;

Since $\tan\frac{B - C}{2} = \frac{b - c}{b + c}\cot\frac{A}{2}$,

$$\therefore \log\left(\tan\frac{B - C}{2}\right) + 10$$
$$= \log(b - c) - \log(b + c) + \log\left(\cot\frac{A}{2}\right) + 10,$$

or, $L\tan\frac{B - C}{2} = \log(b - c) - \log(b + c) + L\cot\frac{A}{2}$.

Also, since
$$a = \frac{b \,.\, \sin A}{\sin B},$$

$\therefore$ $\log a = \log b + L\sin A - L\sin B$, as in Case III.

Example. Given $b=456\cdot 12$ chains, $c=296\cdot 86$ chains, and $A=74^\circ\,20'$, find the other angles.

Here, $b-c=159\cdot 26$, $b+c=752\cdot 98$.

From the Table we find,

$$\log 159\cdot 26=2\cdot 2021067, \text{ and } \log 752\cdot 98=2\cdot 8767834,$$

$$L\cot 37^\circ\,10'=10\cdot 1202593;$$

$$\therefore L\tan\frac{B-C}{2}=2\cdot 2021067-2\cdot 8767834+10\cdot 1202593$$

$$=9\cdot 4455826=L\tan 15^\circ\,35'\,18''.$$

$$\therefore B-C=31^\circ 10'\,36'', \text{ and } B+C=180^\circ-74^\circ\,20'.$$

Thus $\qquad B+C=105^\circ\,40'$;

$$\therefore 2B=136^\circ\,50'\,36''; \quad 2C=74^\circ\,29'\,24'',$$

or, $\qquad B=68^\circ\,25'\,18''$; or, $C=37^\circ\,14'\,42''$.

260. The formula $a^2=b^2+c^2-2bc\cos A$ may be used in simple cases; or it may be adapted to logarithmic calculation by the use of a subsidiary angle. [Cf. Example LXXVI. (20), p. 256.]

Example. If $b=35$ feet, $c=21$ feet, and $A=50^\circ$, find a, given that $\cos 50^\circ=\cdot 643$.

Here $\qquad a^2=35^2+21^2-2\times 35\times 21\times\cos 50^\circ$;

$$\therefore \frac{a^2}{7^2}=5^2+3^2-2\times 5\times 3\times\cos 50^\circ,$$

$$=25+9-30\times\cdot 643, \quad =14\cdot 71.$$

$$\therefore \frac{a}{7}=3\cdot 82 \text{ nearly}; \quad \text{or, } a=26\cdot 74=\text{about } 26\tfrac{3}{4} \text{ feet.}$$

EXAMPLES. LXVIII.

(1) Find B and C, having given that $A=40^\circ$, $b=131$, $c=72$.

$\log 5\cdot 9=\cdot 7708520$, $L\cot 20^\circ=10\cdot 4389341$,

$\log 2\cdot 03=\cdot 3074960$, $L\tan 38^\circ\,36'=9\cdot 9021604$,

$L\tan 38^\circ\,37'=9\cdot 9024195$.

(2) Find A and B, having given that $a=35$ feet, $b=21$ feet, $C=50^\circ$. log 2 = ·301030, $L \tan 28^\circ 11' = 9·729020$,

$L \tan 65^\circ = 10·331327$, $L \tan 28^\circ 12' = 9·729323$.

(3) If $b=19$ chains, $c=20$ chains, $A=60^\circ$, find B and C, having given that log 3·9 = ·591065, $L \tan 2^\circ 32' = 8·645853$,

$L \cot 30^\circ = 10·238561$, $L \tan 2^\circ 33' = 8·648704$.

(4) Given that $a=376·375$ chains, $b=251·765$ chains, and $C=78^\circ 26'$, find A and B. $L \cot 39^\circ 13' = 10·0882755$,

log 1·2461 = ·0955529, $L \tan 13^\circ 39' = 9·3853370$,
log 6·2814 = ·7980565, $L \tan 13^\circ 40' = 9·3858876$.

(5) If $a=135$, $b=105$, $C=60^\circ$, find A, having given that

log 2 = ·3010300, $L \tan 12^\circ 12' = 9·3348711$,
log 3 = ·4771213, $L \tan 12^\circ 13' = 9·3354823$.

(6) If $a=21$ chains, $b=20$ chains, $C=60^\circ$, find c.

(7) Find c in the triangle of example (5).

(8) In a triangle the ratio of two sides is 5 : 3 and the included angle is $76^\circ 30'$. Find the other angles.

log 2 = ·3010300, $L \cot 35^\circ 15' = 10·1507464$,
$L \tan 19^\circ 28' 50'' = 9·5486864$.

Case IV.

261. Given **two sides** and the **angle opposite** one of them, as b, c, B. [Omitted in Euc. I; Euc. VI. 7.]

First, since $\dfrac{c}{\sin C} = \dfrac{b}{\sin B}$; $\therefore$ $\sin C = \dfrac{c \sin B}{b}$.

C must be found from this equation.

When C is known, $A = 180° - B - C$,

and, $$a = \frac{b \sin A}{\sin B}.$$

Which solves the triangle.

262. We remark however that the angle C, found from the **trigonometrical** equation *sin* $C = a$ *given quantity*, where C is an angle of a triangle, has **two** values, one less than 90°, and one greater than 90°. [Art. 234.]

The question arises, Are both these values admissible?

This may be decided as follows:

If B is not less than 90°, C must be less than 90°; and the smaller value for C **only** is admissible.

If B is less than 90° we proceed thus.

1. If b is less than $c \sin B$, then $\sin C$, which $= \frac{c \sin B}{b}$, is greater than 1. This is impossible. Therefore if b is less than $c \sin B$, there is **no** solution whatever.

2. If b is equal to $c \sin B$, then $\sin C = 1$, and therefore $C = 90°$; and there is only **one** value of C, viz. 90°.

3. If b is greater than $c \sin B$, and less than c, then B is less than C, and C may be obtuse or acute. In this case C may have either of the values found from the equation $\sin C = \frac{c \sin B}{b}$. Hence there are **two** solutions, and the triangle is said to be **ambiguous.**

4. If b is equal to or greater than c, then B is equal to or greater than C, so that C must be an acute angle; and the smaller value for C **only** is admissible.

263. The same results may be obtained **geometrically.**

Construction. Draw $AB = c$; make the angle $ABD =$ the given angle B; with centre A and radius $= b$ describe a circle; draw AD perpendicular to BD.

Fig. I.

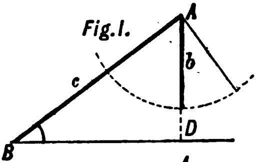


Fig. II.

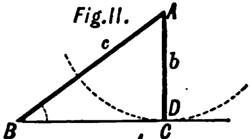


Fig. III.

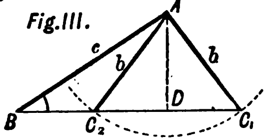
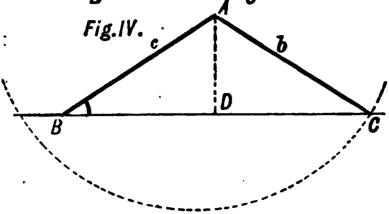


Fig. IV.



Then $AD = c \sin B$.

1. If b is less than $c \sin B$, i.e. less than AD, the circle will not cut BD at all, and the construction **fails.** (Fig. i.)

2. If b is equal to AD, the circle will *touch* the line BD in the point D, and the required triangle is the **right-angled** triangle ABD. (Fig. ii.)

3. If b is greater than AD and less than AB, i.e. than c, the circle will cut the line BD in *two* points C_1, C_2 each on the same side of B. And we get **two** triangles ABC_1, ABC_2 each satisfying the given condition. (Fig. iii.)

4. If b is equal to c, the circle cuts BD in B and in one other point C; if b is greater than c the circle cuts BD in *two* points, but on *opposite* sides of B. In either case there is only **one** triangle satisfying the given condition. (Fig. iv.)

264. We may also obtain the same results **algebraically**, from the formula

$$b^2 = c^2 + a^2 - 2ca\cos B.$$

In this b, c, B are given, a is unknown. Write x for a and we get the **quadratic equation**

$$x^2 - 2c\cos B \, . \, x = b^2 - c^2.$$

Whence, $x^2 - 2c\cos B \, . \, x + c^2\cos^2 B = b^2 - c^2 + c^2\cos^2 B$

$$= b^2 - c^2\sin^2 B;$$

$$\therefore \; x = c\cos B \pm \sqrt{b^2 - c^2\sin^2 B}.$$

Let a_1, a_2 be the two values of x thus obtained, then

$$\left.\begin{aligned} a_1 &= c\cos B + \sqrt{b^2 - c^2\sin^2 B} \\ a_2 &= c\cos B - \sqrt{b^2 - c^2\sin^2 B} \end{aligned}\right\}.$$

Which of these two solutions is admissible, may be decided as follows;

1. If b is less than $c\sin B$, then $(b^2 - c^2\sin^2 B)$ is negative, so that a_1, a_2 are **impossible** quantities.

2. If b is equal to $c\sin B$, then $(b^2 - c^2\sin^2 B) = 0$, and $a_1 = a_2$; thus the *two* solutions become **one**.

3. If b is greater than $c\sin B$, then the **two** values a a_2 are different and positive unless

$$\sqrt{b^2 - c^2\sin^2 B} \quad \text{is} \quad > c\cos B,$$

i.e. unless $\quad b^2 - c^2\sin^2 B \quad > c^2\cos^2 B,$

i.e. unless $\quad b^2 \quad > c^2.$

4. If b is equal to c, then $a_2 = 0$; if b is greater than c, then a_2 is negative and is therefore inadmissible. In either of these cases a_1 is the **only** available solution.

265. We give two examples. In the first there are two solutions, in the second there is only one.

Example 1. Find A and C, having given that $b=379{\cdot}41$ chains, $c=483{\cdot}74$ chains, and $B=34^\circ\,11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}6846120 + 9{\cdot}7496148 - 2{\cdot}5791088 \\ &= 9{\cdot}8551180 = L\sin 45^\circ\,45'; \end{aligned}$$

$$\therefore\ C = 45^\circ\,45', \text{ or, } 180^\circ - 45^\circ\,45' = 134^\circ\,15'.$$

Since b is less than c, each of these values is admissible.

When $C=\ 45^\circ\,45'$, then $A=100^\circ\,4'$.

When $C=134^\circ\,15'$, then $A=11^\circ\,34'$.

Example 2. Find A and C, when $b=483{\cdot}74$ chains, $c=379{\cdot}14$ chains, and $B=34^\circ\,11'$.

$$\begin{aligned} L\sin C &= \log c + L\sin B - \log b \\ &= 2{\cdot}5791088 + 9{\cdot}7496148 - 2{\cdot}6846120 \\ &= 9{\cdot}6441116 = L\sin 26^\circ\,9'; \end{aligned}$$

$$\therefore\ C = 26^\circ\,9', \text{ or, } 180^\circ - 26^\circ\,9' = 153^\circ\,51'.$$

Since b is greater than c, C must be less than 90°, and the larger value for C is inadmissible.

[It is also clear that $(153^\circ\,51' + 34^\circ\,11')$ is $>180^\circ$].

$$\therefore\ C = 26^\circ\,9',\quad A = 119^\circ\,40'.$$

EXAMPLES. LXIX.

(1) If $B=40^\circ$, $b=140{\cdot}5$ feet, $a=170{\cdot}6$ feet, find A and C.

$\log 1{\cdot}405 = {\cdot}1476763$, $L\sin 40^\circ = 9{\cdot}8080675$,

$\log 1{\cdot}706 = {\cdot}2319790$, $L\sin 51^\circ\,18' = 9{\cdot}8923342$,

$L\sin 51^\circ\,19' = 9{\cdot}8924354$.

(2) Find B and C, having given that $A=50^0$, $b=119$ chains, $a=97$ chains, and that

$$\log 1{\cdot}19 = {\cdot}075547,\quad L\sin 50^0 = 9{\cdot}884254,$$
$$\log 9{\cdot}7 = {\cdot}986772,\quad L\sin 70^0 = 9{\cdot}972986,$$
$$L\sin 70^0\, 1' = 9{\cdot}973032.$$

(3) Find B, C, and c, having given that $A=50^0$, $b=97$, $a=119$ (see example (2)).

$$\log 1{\cdot}553 = {\cdot}191169,\quad L\sin 38^0\, 38'\, 24'' = 9{\cdot}795479,$$
$$L\sin 88^0\, 37'\, 24'' = 9{\cdot}999876.$$

(4) Find A, having given that $a=24$, $c=25$, $C=65^0\, 59'$, and that

$$\log 2{\cdot}5 = {\cdot}3979400,\quad L\sin 65^0\, 59' = 9{\cdot}9606739,$$
$$\log 2{\cdot}4 = {\cdot}3802112,\quad L\sin 61^0\, 16' = 9{\cdot}9429335,$$
$$L\sin 61^0\, 17' = 9{\cdot}9430028.$$

(5) If $a=25$, $c=24$, and $C=65^0\, 59'$, find A, B and the greater value of b.

$$\log 1{\cdot}755 = {\cdot}2442771,\quad L\sin 72^0\, 4' = 9{\cdot}9783702,$$
$$\log 1{\cdot}756 = {\cdot}2445245,\quad L\sin 72^0\, 5' = 9{\cdot}9784111,$$
$$L\sin 41^0\, 56'\, 10'' = 9{\cdot}8249725,$$
$$L\sin 41^0\, 56'\, 26'' = 9{\cdot}8249959\,;$$

for other logs see example (4).

(6) Supposing the data for the solution of a triangle to be as in the three following cases (α), (β), (γ), point out whether the solution will be ambiguous or not, and find the third side in the obtuse-angled triangle in the ambiguous case :

(α) $A=30^0$, $a=125$ feet, $c=250$ feet,
(β) $A=30^0$, $a=200$ feet, $c=250$ feet,
(γ) $A=30^0$, $a=200$ feet, $c=125$ feet.

$$\log 2 = {\cdot}3010300,\quad L\sin 38^0\, 41' = 9{\cdot}7958800,$$
$$\log 6{\cdot}0389 = {\cdot}7809578,\quad L\sin 8^0\, 41' = 9{\cdot}1789001,$$
$$\log 6{\cdot}0390 = {\cdot}7809650.$$

266. It saves a little trouble in practice when using the formula $a=\frac{b \sin A}{\sin B}$, to write it thus; $a=b \sin A \,.\, \operatorname{cosec} B$.

For then, $\log a = \log b + L \sin A + L \operatorname{cosec} B - 20$.

Thus the *subtraction* of a logarithm is avoided.

* 267. In the following Examples the student must find the necessary logarithms etc. from the Tables.

*MISCELLANEOUS EXAMPLES. LXX.

(1) Find A when $a=374{\cdot}5$, $b=576{\cdot}2$, $c=759{\cdot}3$ feet.

(2) Find B when $a=4001$, $b=9760$, $c=7942$ yards.

(3) Find C when $a=8761{\cdot}2$, $b=7643$, $c=4693{\cdot}8$ chains.

(4) Find B when $A=86^\circ\ 19'$, $b=4930$, $c=5471$ chains.

(5) Find C when $B=32^\circ\ 58'$, $c=1873{\cdot}5$, $a=764{\cdot}2$ chains.

(6) Find c when $C=108^\circ\ 27'$, $a=36541$, $b=89170$ feet.

(7) Find c when $B=74^\circ\ 10'$, $C=62^\circ\ 45'$, $b=3720$ yards.

(8) Find b when $B=100^\circ\ 19'$, $C=44^\circ\ 59'$, $a=1000$ chains.

(9) Find a when $B=123^\circ\ 7'\ 20''$, $C=15^\circ\ 9'$, $c=9964$ yards.

Find the other two angles in the six following triangles.

(10) $C=100^\circ\ 37'$, $b=1450$, $c=6374$ chains.

(11) $C=52^\circ\ 10'$, $b=643$, $c=872$ chains.

(12) $A=76^\circ\ 2'\ 30''$, $b=1000$, $a=2000$ chains.

(13) $C=54^\circ\ 23'$, $b=873{\cdot}4$, $c=752{\cdot}8$ feet.

(14) $C=18^\circ\ 21'$, $b=674{\cdot}5$, $c=269{\cdot}7$ chains.

(15) $A=29^\circ\ 11'\ 43''$, $b=7934$, $a=4379$ feet.

(16) The difference between the angles at the base of a triangle is $17^\circ\ 48'$, and the sides subtending those angles are $105{\cdot}25$ feet and $76{\cdot}75$ feet; find the third angle.

(17) If $b : c = 4 : 5$, $a=1000$ yards and $A=37^\circ\ 19'$, find b.

The student will find some Examples of Solution of Triangles without the aid of logarithms, in an Appendix.

CHAPTER XVIII.

ON THE MEASUREMENT OF HEIGHTS AND DISTANCES.

268. We have said (Art. 97) that the measurement, with scientific accuracy, of a line of any considerable length involves a long and difficult process.

On the other hand, sometimes it is required to find the *direction* of a line that it may point to an object which is not visible from the point from which the line is drawn. As for example when a tunnel has to be constructed.

By the aid of the Solution of Triangles

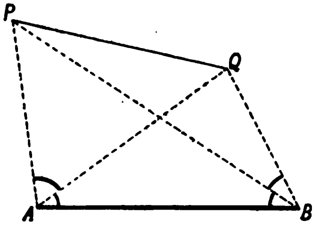
we can find the length of the distance between points which are inaccessible;

we can calculate the magnitude of angles which cannot be practically observed;

we can find the relative heights of distant and inaccessible points.

The method on which the Trigonometrical Survey of a country is conducted affords the following illustration.

269. *To find the distance between two distant objects.*



Two convenient positions A and B, on a level plain as far apart as possible, having been selected, the distance between A and B is measured with the greatest possible care. This line AB is called the **base** line. (In the survey of England, the base line is on Salisbury plain, and is about 36,578 feet long).

Next, the two distant objects, P and Q (church spires, for instance) visible from A and B, are chosen.

The angles PAB, PBA are observed. Then by Case II. Chapter XVII, the lengths of the lines PA, PB are calculated.

Again, the angles QAB, QBA are observed; and by Case II. the lengths of QA and QB are calculated.

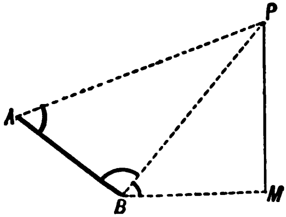
Thus the lengths of PA and QA are found.

The angle PAQ is observed; and then by Case III. the length of PQ is calculated.

270. Thus the distance between two points P and Q has been found. The points P and Q are not necessarily accessible; the only condition being that P and Q must be *visible* from both A and B.

271. In practice, the points P and Q will generally be accessible, and then the line PQ, whose length has been calculated, may be used as a new base to find other distances.

272. *To find the height of a distant object above the point of observation.*



Let B be the point of observation; P the distant object. From B measure a base line BA of any convenient length, in any convenient direction; observe the angles PAB, PBA, and by Case II. calculate the length of BP. Next observe at B the 'angle of elevation' of P; that is, the angle which the line BP makes with the horizontal line BM, M being the point in which the vertical line through P cuts the horizontal plane through B.

Then PM, which is the vertical height of P above B can be calculated, for $PM = BP . \sin MBP$.

Example 1. *The distance between a church spire* A *and a milestone* B *is known to be* 1764·3 *feet; C is a distant spire. The angle* CAB *is* 94° 54′, *and the angle* CBA *is* 66° 39′. *Find the distance of* C *from* A.

ABC is a triangle and we know one side c and two angles (A and B), and therefore it can be solved by Case II.

$$\text{The angle } ACB = 180^\circ - 94^\circ\, 54' - 66^\circ\, 39'$$
$$= 18^\circ\, 27'.$$

Therefore the triangle is the same as that solved on page 210. Therefore $AC = 5118{\cdot}2$ feet.

Example 2. *If the spire* C *in the last Example stands on a hill, and the angle of elevation of its highest point is observed at* A *to be* $4^0\ 19'$; *find how much higher* C *is than* A.

The required height $x = AC \,.\, \sin 4^0\ 19'$ [Art. 268], and AC is 5118·2 feet,

$$\begin{aligned} \therefore \log x &= \log (AC \,.\, \sin 4^0\ 19') \\ &= \log 5118{\cdot}2 + L \sin 4^0\ 19' - 10 \\ &= 3{\cdot}7091173 + 8{\cdot}8766150 - 10 \\ &= 2{\cdot}5857323 = \log 385{\cdot}24. \end{aligned}$$

Therefore $x = 385$ ft. 3 in. nearly.

EXAMPLES. LXXI.

(Exercises XIV. and LXI. consist of easy examples on this subject).

(1) Two straight roads inclined to one another at an angle of 60^0, lead from a town A to two villages B and C; B on one road distant 30 miles from A, and C on the other road distant 15 miles from A. Find the distance from B to C. Ans. 25·98 m.

(2) Two ships leave harbour together, one sailing N.E. at the rate of $7\frac{1}{2}$ miles an hour and the other sailing North at the rate of 10 miles an hour. Prove that the distance between the ships after an hour and a half is 10·6 miles.

(3) A and B are two consecutive milestones on a straight road and C is a distant spire. The angles ABC and BAC are observed to be 120^0 and 45^0 respectively. Show that the distance of the spire from A is 3·346 miles.

(4) If the spire C in the last question stands on a hill, and its angle of elevation at A is 15^0, show that it is ·866 of a mile higher than A.

(5) If in Question (3) there is another spire D such that the angles DBA and DAB are 45^0 and 90^0 respectively and the angle DAC is 45^0; prove that the distance from C to D is $2\frac{3}{4}$ miles very nearly.

(6) A and B are two consecutive milestones on a straight road, and C is the chimney of a house visible from both A and B. The angles CAB and CBA are observed to be $36^0\,18'$ and $120^0\,27'$ respectively. Show that C is 2639·5 yards from B,

log 1760 $=$ 3·2455127 $\quad$ L sin $36^0\,18' =$ 9·7723314
log 2639·5 $=$ 3·42152 $\quad$ L cosec $23^0\,15' =$ 10·4036846.

(7) A and B are two points on opposite sides of a mountain, and C is a place visible from both A and B. It is ascertained that C is distant 1794 feet and 3140 feet from A and B respectively and the angle ACB is $58^0\,17'$. Show that the angle which the line pointing from A to B makes with AC is $86^0\,55'\,49''$,

log 1346 $=$ 3·1290451 $\quad$ L cot $29^0\,8'\,30'' =$ 10·2537194
log 4934 $=$ 3·6931991 $\quad$ L tan $26^0\,4'\,19'' =$ 9·6895654

(8) A and B are two hill-tops 34920 feet apart, and C is the top of a distant hill. The angles CAB and CBA are observed to be $61^0\,53'$ and $76^0\,49'$ respectively. Prove that the distance from A to C is 51515 feet,

log 34920 $=$ 4·5430742 $\quad$ L sin $76^0\,49' =$ 9·9884008
log 51515 $=$ 4·71193 $\quad$ L cosec $41^0\,18' =$ 10·1804552.

(9) From two stations A and B on shore, 3742 yards apart, a ship C is observed at sea. The angles BAC, ABC are simultaneously observed to be $72^0\,34'$ and $81^0\,41'$ respectively. Prove that the distance from A to the ship is 8522·7 yards,

log 3742 $=$ 3·5731038 $\quad$ L sin $81^0\,41' =$ 9·9954087
log 8522·7 $=$ 3·9305774 $\quad$ L cosec $25^0\,45' =$ 10·3620649.

(10) The distance between two mountain peaks is known to be 4970 yards, and the angle of elevation of one of them when seen from the other is $9^0\,14'$. How much higher is the first than the second? Sin $9^0\,14' =$ ·1604555. *Ans.* 797·5 yards.

(11) Two straight railways intersect at an angle of 60^0. From their point of intersection two trains start, one on each line, one at the rate of 40 miles an hour. Find the rate of the second train that at the end of an hour they may be 35 miles apart. *Ans.* Either 25 or 15 miles an hour. (Art. 264.)

(12) *A* and *B* are two positions on opposite sides of a mountain; *C* is a point visible from *A* and *B*; *AC* and *BC* are 10 miles and 8 miles respectively, and the angle *BCA* is 60^0. Prove that the distance between *A* and *B* is 9·165 miles.

(13) In the last question, if the angle of elevation of *C* at *A* is 8^0, and at *B* is $2^0\ 48'\ 24''$: show that the height of *A* above *B* is one mile very nearly.

$$\sin 8^0 = \cdot 1391731 \quad \sin 2^0\ 48'\ 24'' = \cdot 0489664.$$

(14) Show that the angles which a tunnel going through the mountain from *A* to *B*, in Questions (12) and (13), would make (i) with the horizon, (ii) with the line joining *A* and *C*, are respectively $6^0\ 16'$ and $49^0\ 6'\ 24''$.

$$\sin 6^0\ 16' = \cdot 1091\ ; \quad \tan 10^0\ 53'\ 36'' = \cdot 192450.$$

(15) *A* and *B* are consecutive milestones on a straight road; *C* is the top of a distant mountain. At *A* the angle *CAB* is observed to be $38^0\ 19'$; at *B* the angle *CBA* is observed to be $132^0\ 42'$, and the angle of elevation of *C* at *B* is $10^0\ 15'$. Show that the top of the mountain is 1243·5 yards higher than *B*.

$L \sin 38^0\ 19' = 9\cdot 7923968$

$L \operatorname{cosec} 8^0\ 59' = 10\cdot 8064659$

$L \sin 10^0\ 15' = 9\cdot 2502822.$

$\log 1760 = 3\cdot 2455127$

$\log 1243\cdot 5 = 3\cdot 09465$

(16) A base line *AB*, 1000 feet long is measured along the straight bank of a river; *C* is an object on the opposite bank; the angles *BAC*, and *CBA* are observed to be $65^0\ 37'$ and $53^0\ 4'$ respectively. Prove that the perpendicular breadth of the river at *C* is 829·87 feet; having given

$L \sin 65^0\ 37' = 9\cdot 9594248,$

$L \operatorname{cosec} 61^0\ 19' = 10\cdot 0568589,$

$L \sin 53^0\ 4' = 9\cdot 9027289$

$\log 8\cdot 2987 = \cdot 91901.$

* MISCELLANEOUS EXAMPLES. LXXII.

(1) A man walking along a straight road at the rate of three miles an hour sees, in front of him at an elevation of 60^0, a balloon which is travelling horizontally in the same direction at the rate of six miles an hour; ten minutes after he observes that the elevation is 30^0. Prove that the height of the balloon above the road is $440\sqrt{3}$ yards.

(2) A person standing at a point A, due south of a tower built on a horizontal plain, observes the altitude of the tower to be 60^0. He then walks to a point B due west from A and observes the altitude to be 45^0, and then at the point C in AB produced he observes the altitude to be 30^0. Prove that $AB=BC$.

(3) The angle of elevation of a balloon, which is ascending uniformly and vertically, when it is one mile high is observed to be $35^0\,20'$; 20 minutes later the elevation is observed to be $55^0\,40'$. How fast is the balloon moving?

Ans. $3(\sin 20^0\,20')(\sec 55^0\,40')(\operatorname{cosec} 35^0\,20')$ miles per hour.

(4) The angular elevation of a tower at a place A due south of it is 30^0; and at a place B due west of A, and at a distance a from it, the elevation is 18^0; show that the height of the tower is

$$\frac{a}{\sqrt{2(1+\sqrt{5})}}.$$

(5) The angular elevation of the top of a steeple at a place due south of it is 45^0, and at another place due west of the former station and distant a feet from it the elevation is 15^0; show that the height of the steeple is $\frac{a}{2}(3^{\frac{1}{4}}-3^{-\frac{1}{4}})$ feet.

(6) A tower stands at the foot of an inclined plane whose inclination to the horizon is 9^0; a line is measured up the incline from the foot of the tower of 100 feet in length. At the upper extremity of this line the tower subtends an angle of 54^0. Find the height of the tower. *Ans.* 114·4 ft.

(7) The altitude of a certain rock is observed to be 47^0, and after walking 1000 feet towards the rock, up a slope inclined at an angle of 32^0 to the horizon, the observer finds that the altitude is 77^0. Prove that the vertical height of the rock above the first point of observation is 1034 ft. $\sin 47^0 = \cdot 73135$.

(8) At the top of a chimney 150 feet high standing at one corner of a triangular yard, the angle subtended by the adjacent sides of the yard are 30^0 and 45^0 respectively; while that subtended by the opposite side is 30^0. Show that the lengths of the sides are 150 ft. 86·6 ft. and 106 ft. respectively.

(9) A flag staff h feet high stands on the top of a tower. From a point in the plain on which the tower stands the angles of elevation of the top and bottom of the flagstaff are observed to be a and β respectively. Prove that the height of the tower is $\frac{h \tan \beta}{\tan a - \tan \beta}$ feet, i.e. $\frac{h \sin \beta . \cos a}{\sin (a - \beta)}$ feet.

(10) From the top of a cliff h feet high the angles of depression of two ships at sea in a line with the foot of the cliff are a and β respectively. Show that the distance between the ships is h (cot $-$ cot a) feet.

(11) The angular elevation of a tower at a place due south of it is a, and at another place due west of the first and distant d from it, the elevation is β. Prove that the height of the tower is

$$\frac{d}{\sqrt{\cot^2 \beta - \cot^2 a}}, \text{ i.e. } \frac{d \sin a . \sin \beta}{\sqrt{\sin (a - \beta) . \sin (a + \beta)}}.$$

(12) A man stands on the top of the wall of height h, and observes the angular elevation (a) of the top of a telegraph post; he then descends from the wall, and finds that the angular elevation is now β; prove that the height of the post exceeds the height of the man by $h \frac{\sin \beta . \cos a}{\sin (\beta - a)}$.

(13) Find the height of a cloud by observing (i) its elevation (α) and (ii) the depression (β) of its reflexion in a lake h feet below the point of observation. *Ans.* Height above the lake $h \sin(\alpha+\beta) \operatorname{cosec}(\beta-\alpha)$ feet.

(14) If the angular elevation of the summits of two spires (which appear in a straight line) is α, and the angular depressions of their reflexions in a lake, h feet below the point of observation, are β and γ, then the horizontal distance between the spires is $2h \cos^2 \alpha \sin(\beta-\gamma) . \operatorname{cosec}(\beta-\alpha) . \operatorname{cosec}(\gamma-\alpha)$ feet.

(15) A statue AB on the top of a pillar BC situated on level ground is found to subtend the greatest angle (α) at the eye of an observer E when he is distant c feet from the pillar. Prove that the height of the statue is ($2\,c \tan \alpha$) feet; and, if b be the height of the observer's eye from the ground find the height of the pillar. (See Note in the Answers.)

(16) A man walking along a straight road observes that the greatest angle which two objects subtend at his eye is α. From the place where this happens he walks a yards; the objects then appear in a straight line making a right angle with the road. Prove that the distance between the objects is ($2\,a \tan \alpha$) yards.

(17) A man standing on a horizontal plain at a distance h feet from a tower, observes that a flagstaff on the tower subtends an angle α at his eye, and that on walking $2\,k$ feet towards the tower the flagstaff subtends again the same angle. Prove that the height of the flagstaff is $2\,(h-k) \tan \alpha$ feet.

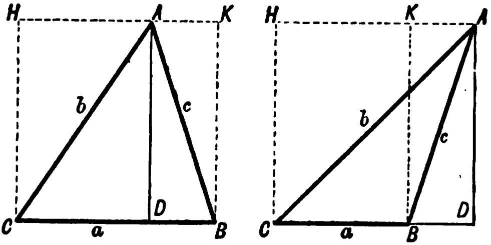
(18) A man walking along a straight road observes that the greatest angle which two objects subtend at his eye is α; from the place where this happens he walks a yards, and the objects then appear in a straight line making an angle β with the road. Prove that the distance between the objects is $\dfrac{2a \sin \alpha . \sin \beta}{\cos \alpha + \cos \beta}$ yds.

CHAPTER XIX.

ON TRIANGLES AND CIRCLES.

273. *To find the* **Area** *of a Triangle.*

The area of the triangle ABC is denoted by Δ.



Through A draw HK parallel to BC, and through ABC draw lines AD, BK, CH perpendicular to BC.

The area of the triangle ABC is half that of the rectangular parallelogram $BCHK$ [Euc. I. 41].

Therefore
$$\Delta = \frac{BC \,.\, CH}{2} = \frac{BC \,.\, DA}{2}$$

$$= \frac{a \,.\, b \sin C}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

But
$$\sin C = \frac{2}{ab} \,.\, \sqrt{s(s-a)(s-b)(s-c)}\,;$$

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)} = S \ldots\ldots \text{(ii)}.$$

274. *To find the Radius of the* **Circumscribing Circle.**

Fig. I.

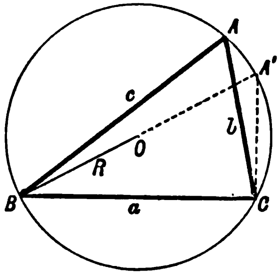
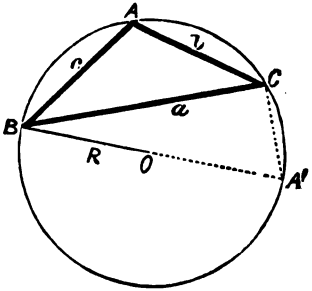


Fig. II.



Let a circle $AA'CB$ be described about the triangle ABC. Let R stand for its radius. Let O be its centre. Join BO, and produce it to cut the circumference in A'. Join $A'C$.

Then, Fig. I. the angles BAC, $BA'C$ in the same segment are equal; Fig. II. the angles BAC, $BA'C$ are supplementary; also the angle BCA' in a semicircle is a right angle.

Therefore $\dfrac{CB}{A'B} = \sin CA'B = \sin CAB = \sin A$,

or,
$$\frac{a}{2R} = \sin A\,;$$

$$\therefore\ 2R = \frac{a}{\sin A}.$$

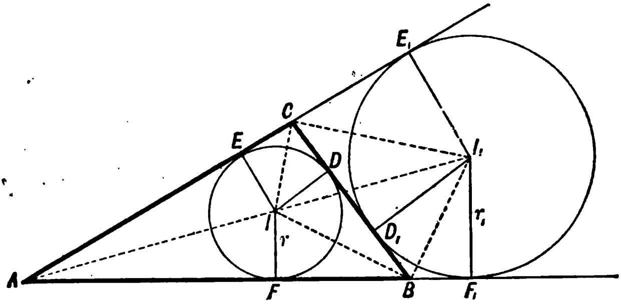
275. Similarly, it may be proved that

$$2R = \frac{b}{\sin B}\,;\ \text{and that } 2R = \frac{c}{\sin C}.$$

Hence,
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Thus d, the value of each of these fractions, is the diameter of the circumscribing circle.

276. *To find the radius of the* **Inscribed Circle.**



Let D, E, F be the points in which the circle inscribed in the triangle ABC touches the sides. Let I be the centre of the circle; let r be its radius. Then $ID = IE = IF = r$.

The area of the triangle ABC

$$= \text{area of } IBC + \text{area of } ICA + \text{area of } IAB.$$

And the area of the triangle $IBC = \frac{1}{2} ID \,.\, BC = \frac{1}{2} r \,.\, a$,

$$\therefore \text{ area of } ABC = \tfrac{1}{2} ID \,.\, BC + \tfrac{1}{2} IE \,.\, CA + \tfrac{1}{2} IF \,.\, AB$$

$$= \tfrac{1}{2} ra + \tfrac{1}{2} rb + \tfrac{1}{2} rc;$$

or,

$$\Delta = \tfrac{1}{2} r (a + b + c) = \tfrac{1}{2} r \,.\, 2s = rs.$$

$$\therefore r = \frac{\Delta}{s} = \frac{S}{s}.$$

277. A circle which touches one of the sides of a triangle and the other two sides produced is called an **Escribed Circle** of the triangle.

278. *To find the radius of an* **Escribed Circle.**

Let an escribed circle touch the side BC and the sides AC, AB produced in the points D_1, E_1, F_1 respectively. Let I_1 be its centre, r_1 its radius. Then

$$I_1D_1 = I_1E_1 = I_1F_1 = r_1.$$

The area of the triangle ABC

$$= \text{area of } ABI_1C - \text{area of } I_1BC,$$

$$= \text{area of } I_1CA + \text{area of } I_1AB - \text{area of } I_1BC,$$

or
$$\Delta = \tfrac{1}{2}I_1E_1 \,.\, CA + \tfrac{1}{2}I_1F_1 \,.\, AB - \tfrac{1}{2}I_1D_1 \,.\, BC$$

$$= \tfrac{1}{2}r_1b + \tfrac{1}{2}r_1c - \tfrac{1}{2}r_1a$$

$$= \tfrac{1}{2}r_1(b + c - a) = \tfrac{1}{2}r_1(2s - 2a)$$

$$= r_1(s - a).$$

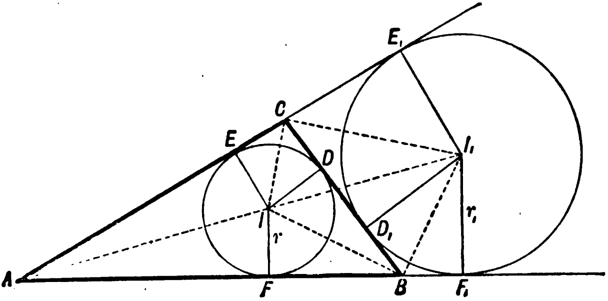
$$\therefore r_1 = \frac{\Delta}{s-a} = \frac{S}{s-a}.$$

279. Similarly if r_2 and r_3 be the radii of the other two escribed circles of the triangle ABC, then

$$r_2 = \frac{S}{s-b}; \quad r_3 = \frac{S}{s-c}.$$

*280. The following results are often useful.

With the construction of the last two articles; the lines IA, IB, IC bisect the angles A, B, C respectively; the lines I_1A, I_1B, I_1C bisect the angles BAC, CBF_1, BCE_1 respectively; so that AII_1 is a straight line.



(i) Two tangents drawn from an external point to touch a circle are equal; therefore

$$AE = AF;\quad BD = BF;\quad CE = CD;$$
$$AE_1 = AF_1;\ BD_1 = BF_1;\ CE_1 = CD_1.$$

(ii) $2AE_1 = AE_1 + AF_1 = (AC + CD_1) + (AB + BD_1)$

$$= AC + CB + BA = a + b + c = 2s.$$

(iii) $2AE = AE + AF = (AC - CD) + (AB - BD)$

$$= AC + AB - BC = b + c - a = 2(s - a).$$

Similarly, $BF = BD = (s - b)$; $CD = CE = (s - c)$.

(iv) $BD_1 = BF_1 = AF_1 - AB = (s - c)$.

Similarly, $CE_1 = CD_1 = (s - b)$.

(v) Hence, $BD_1 = CD$, and $CD_1 = BD$.

(vi) $FF_1 = AF_1 - AF = s - (s - a) = a$.

Similarly, $EE_1 = a$.

(vii) $II_1 = AI_1 - AI = AF_1 \sec\frac{A}{2} - AF\sec\frac{A}{2} = FF_1 \sec\frac{A}{2}$

$$\therefore\ II_1 = \frac{a}{\cos\frac{A}{2}}.$$

(viii) $IF = AF\tan\frac{A}{2}$; or, $r = (s-a)\tan\frac{A}{2}$,

$I_1F = AF_1\tan\frac{A}{2}$; or, $r_1 = s\tan\frac{A}{2}$.

(ix) $AF = IF\cot\frac{A}{2}$, and $BF = IF\cot\frac{B}{2}$;

$$\therefore\ c = AF + BF = r\left(\cot\frac{A}{2} + \cot\frac{B}{2}\right);$$

$$\therefore\ r = \frac{c}{\cot\frac{A}{2} + \cot\frac{B}{2}} = \frac{c\sin\frac{A}{2}.\sin\frac{B}{2}}{\sin\frac{A+B}{2}} = \frac{c\sin\frac{A}{2}.\sin\frac{B}{2}}{\cos\frac{C}{2}}.$$

(x) $F_1BC = 180^\circ - B$, $\therefore I_1BC = 90^\circ - \frac{B}{2}$. Hence,

$$r_1\left\{\cot\left(90^\circ - \frac{B}{2}\right) + \cot\left(90^\circ - \frac{C}{2}\right)\right\} = a;$$

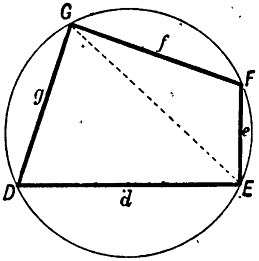
or,

$$r_1 = \frac{a\cos\frac{B}{2}.\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

(xi) $r_1 = AF_1\tan\frac{A}{2}$, and $r = AF\tan\frac{A}{2}$; therefore

$$r_1 - r = FF_1\tan\frac{A}{2} = a\tan\frac{A}{2}.$$

*281. *To find an expression for the* **area** *of a quadrilateral inscribed in a circle, in terms of the sides.*



Let $DEFG$ be the quadrilateral; and let d, e, f, g represent the sides.

The angle $F = 180° - D$. [Euc. III. 22.]

The area of $GDE = \frac{1}{2}dg \sin D$. [Art. 273.]

The area of $EFG = \frac{1}{2}ef \sin F = \frac{1}{2}ef \sin (180° - D)$

$$= \frac{1}{2}ef \sin D.$$

Thus the area of the quadrilateral is

$$\frac{1}{2}(dg + ef) \sin D.$$

We must find an expression for $\sin D$ in terms of the sides.

From the triangle EGD we have

$$EG^2 = d^2 + g^2 - 2dg \cos D. \qquad \text{[Art. 239.]}$$

From the triangle EGF we have

$$EG^2 = e^2 + f^2 - 2ef \cos F$$

$$= e^2 + f^2 + 2ef \cos D; \qquad \text{[For } F = 180° - D\text{]}$$

$$\therefore \; d^2 + g^2 - 2dg \cos D = e^2 + f^2 + 2ef \cos D;$$

$$\therefore\ 2(dg+ef)\cos D = d^2+g^2-e^2-f^2;$$

$$\therefore\ 1-\cos^2 D = 1-\frac{(d^2+g^2-e^2-f^2)^2}{4(dg+ef)^2};$$

$$\therefore\ \sin^2 D = \frac{4(dg+ef)^2-(d^2+g^2-e^2-f^2)^2}{4(dg+ef)^2}.$$

The numerator of this expression is equal to

$$\{(2dg+2ef)+(d^2+g^2-e^2-f^2)\}\{(2dg+2ef)-(d^2+g^2-e^2-f^2)\}$$

$$=\{(d+g)^2-(e-f)^2\}\times\{(e+f)^2-(d-g)^2\}$$

$$=\{(d+g)+(e-f)\}\{(d+g)-(e-f)\}$$
$$\times\{(e+f)+(d-g)\}\{(e+f)-(d-g)\}$$

$$=(d+g+e-f)(d+g-e+f)(e+f+d-g)(e+f+g-d).$$

Let 2s stand for $(d+e+f+g)$.

Then the above expression may be written

$$(2s-2f)(2s-2e)(2s-2g)(2s-2d).$$

Therefore $\sin^2 D = \dfrac{4(s-d)(s-e)(s-f)(s-g)}{(dg+ef)^2}$.

or $$\sin D = \frac{2\sqrt{(s-d)(s-e)(s-f)(s-g)}}{dg+ef}.$$

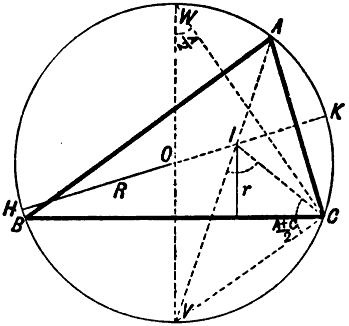
Hence the **area** of the quadrilateral *DEFG*

which is $\frac{1}{2}(dg+ef)\sin D$

$$=\sqrt{(s-d)(s-e)(s-f)(s-g)}.$$

* 282. *To prove that if* I *and* O *are the centres of the inscribed and circumscribed circles of a triangle, then*

$$OI^2 = R^2 - 2Rr.$$



Let OI produced cut the circumscribing circle in H and K. Join IA, IC; produce IA to cut the circle ABC again in V. Draw the diameter VOW and join CW. Then the angle $VIC = IAC + ICA = \frac{1}{2}A + \frac{1}{2}C = \frac{1}{2}(A + C)$. The angle

$$ICV = ICB + BCV = ICB + BAV = \tfrac{1}{2}C + \tfrac{1}{2}A = \tfrac{1}{2}(A + C).$$

Therefore the angle $VCI = VIC$, and $\therefore VI = VC$.

Now $OK^2 - OI^2 = HI . IK$ [Euc. II. 5]

$= VI . IA = VC . IA.$ [Euc. III. 35.]

But $VC = VW . \sin CWV = 2R . \sin CAV = 2R . \sin \dfrac{A}{2}$,

and $IA = \dfrac{r}{\sin \dfrac{A}{2}}$. [See Figure on page 234.]

Hence $$OK^2 - OI^2 = 2R \sin \frac{A}{2} \times \frac{r}{\sin \dfrac{A}{2}},$$

or $$R^2 - OI^2 = 2Rr.$$

EXAMPLES. LXXIII.

(1) Find the area of the triangle ABC when

(i) $a=4$, $b=10$ feet, $C=30^0$.

(ii) $b=5$, $c=20$ inches, $A=60^0$.

(iii) $c=66\frac{2}{3}$, $a=15$ yards, $B=17^0\ 14'$ [$\sin 17^0\ 14' = \cdot 29626$].

(iv) $a=13$, $b=14$, $c=15$ chains.

(v) $a=10$, the perpendicular from A on $BC=20$ feet.

(vi) $a=625$, $b=505$, $c=904$ yards.

(2) Find the Radii of the Inscribed and each of the Escribed Circles of the triangle ABC when $a=13$, $b=14$, $c=15$ feet.

(3) Show that the triangles in which (i) $a=2$, $A=60^0$; (ii) $b=\frac{2}{3}.\sqrt{3}$, $B=30^0$ can be inscribed in the same circle.

(4) Prove that $R=\dfrac{abc}{4S}$; find R in the triangle of (2).

(5) Prove that if a series of triangles of equal perimeter are described about the same circle, they are equal in area.

(6) If $A=60^0$, $a=\sqrt{3}$, $b=\sqrt{2}$, prove that the area

$$=\tfrac{1}{4}(3+\sqrt{3}).$$

(7) Prove that each of the following expressions represents the area of the triangle ABC:

(i) $\dfrac{abc}{4R}$. (ii) $2R^2 \sin A . \sin B . \sin C$.

(iii) rs. (iv) $Rr(\sin A + \sin B + \sin C)$.

(v) $\frac{1}{2}a^2 \sin B . \sin C . \operatorname{cosec} A$. (vi) $ra \operatorname{cosec} \dfrac{A}{2} \cos \dfrac{B}{2} \cos \dfrac{C}{2}$.

(vii) $(rr_1r_2r_3)^{\frac{1}{2}}$. (viii) $\frac{1}{2}(a^2-b^2) \sin A . \sin B . \operatorname{cosec}(A-B)$.

Prove the following statements:

(8) If a, b, c are in A. P., then $ac=6rR$.

(9) The area of the greatest triangle, two of whose sides are 50 and 60 feet, is 1500 sq. feet.

(10) If the altitude of an isosceles triangle is equal to the base, R is five-eighths of the base.

(11) $R(\sin A+\sin B+\sin C)=s$.

(12) $bc=4R^2(\cos A+\cos B\,.\cos C)$.

(13) If C is a right angle, $2r+2R=a+b$.

(14) $r_2r_3+r_3r_1+r_1r_2=s^2$.

(15) $\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}=\frac{1}{2rR}$.

(16) $r_1\cot\frac{A}{2}=r_2\cot\frac{B}{2}=r_3\cot\frac{C}{2}=r\cot\frac{A}{2}\,.\cot\frac{B}{2}\,.\cot\frac{C}{2}$.

(17) $r_1+r_2=c\cot\frac{C}{2}$.

(18) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r}$.

(19) $\frac{r_1-r}{a}+\frac{r_2-r}{b}=\frac{c}{r_3}$. (20) $r_1+r_2+r_3-r=4R$.

(21) $a\,.\,b\,.\,c\,.\,r=4R(s-a)(s-b)(s-c)$.

(22) The distances of the centres of the escribed circles of the triangle ABC from that of the inscribed circle are

$$4R\sin\frac{A}{2},\ \ 4R\sin\frac{B}{2},\ \ 4R\sin\frac{C}{2}.$$

(23) If A is a right angle, $r_2+r_3=a$.

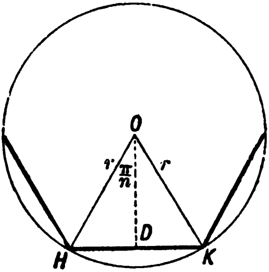
(24) In an equilateral triangle $3R=6r=2r_1$.

(25) $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{1}{r}-\frac{1}{2R}$.

* CHAPTER XX.

ON THE AREA OF THE CIRCLE, THE CONSTRUCTION OF TRIGONOMETRICAL TABLES, &c.

283. Let r be the radius of the circle **described** about a regular polygon of n sides. Let O be the centre; HK one of the sides.



From O draw OD perpendicular to HK, bisecting both HK and the angle HOK. Then, since the polygon has n equal sides, n times the angle HOK make up four right angles

$$\therefore HOK = \frac{2\pi}{n}, \text{ and } \therefore DOK = \frac{\pi}{n}. \quad \text{Hence,}$$

the **perimeter** of the polygon, which is n times HK,

$$= 2n . DH = 2n . OH \sin DOH$$

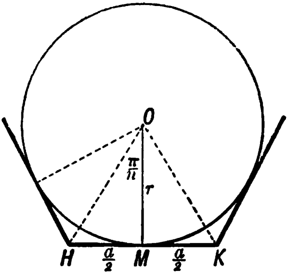
$$= 2nr \sin \frac{\pi}{n};$$

the **area** of the polygon, which is n times the area of HOK,

$$= nOD . HD = nr \cos \frac{\pi}{n} . r \sin \frac{\pi}{n}$$

$$= nr^2 \sin \frac{\pi}{n} . \cos \frac{\pi}{n}.$$

284. Let r be the radius of the circle **inscribed** in a regular polygon of n sides. Let O be the centre, HK one of the sides.



From O draw OM perpendicular to HK, bisecting both HK and the angle HOK. Then since the polygon has n equal sides, n times the angle HOK make up four right angles,

$$\therefore HOK = \frac{2\pi}{n} \text{ and } \therefore MOH = \frac{\pi}{n}. \quad \text{Hence}$$

the **perimeter** of the polygon, which is n times HK,

$$= 2n \,.\, MH = 2n \,.\, OM \tan MOH$$

$$= 2nr \tan \frac{\pi}{n}.$$

The **area** of the polygon, which is n times the area of HOK,

$$= n \,.\, OM \,.\, MH = n \,.\, OM \,.\, OM \tan MOH$$

$$= nr^2 \tan \frac{\pi}{n}$$

$$= \text{the radius} \times \text{half the perimeter.}$$

285. Let a circle of radius r have a regular polygon of n sides circumscribed about it, and also a regular polygon of the same number (n) of sides inscribed in it.

The **perimeter** of the circumscribing polygon is $2nr \tan \frac{\pi}{n}$.

The **perimeter** of the inscribed polygon is $2nr \sin \frac{\pi}{n}$.

The ratio of these perimeters is $1 : \cos \frac{\pi}{n}$ ....(1).

The **area** of the circumscribed polygon is $nr^2 \tan \frac{\pi}{n}$.

The **area** of the inscribed polygon is $nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$.

The ratio of these areas is $1 : \cos^2 \frac{\pi}{n}$ .... (ii).

286. We must assume the two following axioms:

(i) The circumference of the circle lies in magnitude **between** the perimeter of a circumscribed and that of an inscribed polygon.

(ii) The area of the circle lies in magnitude **between** the area of a circumscribed and that of an inscribed polygon.

Now, when n is increased, $\frac{\pi}{n}$ is diminished, and therefore (by Art. 94) $\cos \frac{\pi}{n}$ approaches 1.

Hence, as the number of the sides of the two polygons in Art. 285 is increased, their perimeters approach to equality.

And since the circumference of the circle always lies in magnitude **between** them, each of these perimeters must approach the circumference of the circle.

Therefore, the **circumference** of a circle is that, to which the perimeter of a regular inscribed (or circumscribed) polygon approaches as the number of its sides is increased, and from which it can be made to differ by less than any assignable quantity however small. (Cf. Art. 32.)

287. In like manner it follows, from (ii) Art. 285, that the **area** of the circle is the '*limit*' of the area of a regular inscribed (or circumscribed) polygon when the number of the sides is indefinitely increased.

Now, twice the area of a polygon circumscribed about a circle is equal to (the radius × the perimeter). [Art. 284]

This is true however great be the number of the sides. It is therefore true when the number of the sides is indefinitely increased.

Therefore it is true for the circle itself. Hence,

twice the area of a circle = the radius × the circumference.

Or, the **area** of a circle $= \frac{1}{2}r\,(2\pi r)$ [Art. 34]

$$= \pi r^2.$$

EXAMPLES. LXXIV.

(1) Prove the surd expressions of Art. 37 for the ratios of the perimeters of certain regular polygons to the diameters of their circumscribing circles.

(2) Prove that the area of a regular polygon of twelve sides described about a circle whose radius is 1 foot is 3·215 sq. ft.

(3) Prove that the area of a square described about a circle is $\frac{4}{3}$ of the dodecagon inscribed in the same circle.

(4) Find the perimeter of a regular polygon of 100 sides (i) when described about a circle of 1 foot diameter, (ii) when inscribed in the same circle. *Ans.* (i) 3·14263, (ii) 3·14108 ft.

(5) An equilateral triangle and a regular hexagon have the same perimeter: show that the areas of their inscribed circles are as 4 : 9.

(6) Prove that the area of a regular polygon of n sides, each of whose sides is a, is $\frac{na^2}{4} . \cot \frac{180^0}{n}$.

(7) If the areas of a regular pentagon and decagon are equal, the ratio of their sides is $\sqrt[4]{20} : 1$.

(8) If a be a side of a regular polygon and R and r the radii of the inscribed and described circles respectively, then $2R = a \operatorname{cosec} \frac{\pi}{n}$ and $2r = a \cot \frac{\pi}{n}$.

(9) If R and r be the radii of the inscribed and circumscribed circles of a regular polygon of n sides, each $= a$,

$$R + r = \frac{a}{2} \cot \frac{\pi}{2n}.$$

(10) The triangle formed from one side each of a regular pentagon, hexagon and decagon inscribed in the same circle, is right-angled.

(11) Prove that the area of an irregular polygon described about a circle is equal to the product of the radius and half the perimeter of the polygon.

(12) The area of an irregular polygon of an even number of sides circumscribed about a circle is equal to the radius × the sum of every alternate side.

(13) The diameter of the dome of St Paul's is 108 feet; prove that it covers an area of 1018 sq. yds.

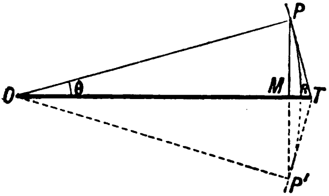
(14) The radius of the circle whose area is one acre is 39¼ yds.

(15) A length of 300 yards of paper, the thickness of which is the hundred and fiftieth part of an inch, is rolled up into a solid cylinder. Find approximately the diameter of the cylinder. *Ans.* 9·575 in.

(16) The diameter of a roll of carpet is 2 feet and the thickness of the carpet is the eighth of an inch. What is the length of the carpet? *Ans.* 301·6 ft.

ON THE CONSTRUCTION OF TRIGONOMETRICAL TABLES.

288. *To prove that, if* θ *be the circular measure of an angle less than a right angle,* $\sin\theta$, θ, $\tan\theta$ *are in ascending order of magnitude.*



Let *ROP* be an angle (θ) less than a right angle. Make the angle *ROP′* on the other side of *OR* equal to *ROP*. With centre *O* and any radius *OR* describe the arc *P′RP*. Draw *PT*, *P′T* to touch the circle at *P* and *P′*. *PT* and *P′T* will meet on *OR*. The line *PP′* is bisected by *OR* at right angles at *M*.

Then, since the circumference of a circle lies *between* the perimeter of the inscribed and circumscribed polygons, it follows that the arc *P′RP* lies in magnitude between *P′MP* and $P'T + TP$. In other words, *P′MP*, the arc *P′RP*, and $P'T + TP$ are in ascending order of magnitude.

Therefore also their halves *MP*, *RP*, *TP* are in ascending order of magnitude.

And so also are $\dfrac{MP}{OP}, \dfrac{RP}{OP}, \dfrac{TP}{OP}$.

That is, $\sin\theta$, θ, $\tan\theta$ are in ascending order of magnitude, where $\theta \left[= \dfrac{\text{arc}}{\text{radius}}\right]$ is the circular measure of the angle referred to.

289. Hence, 1, $\frac{\theta}{\sin\theta}$, $\frac{1}{\cos\theta}$ are in ascending order of magnitude.

290. *To prove that the 'limit' of* $\frac{\theta}{\sin\theta}$, *when* $\theta = 0$, *is* 1; θ *being the circular measure of the angle referred to.*

The value of $\frac{\theta}{\sin\theta}$ lies between 1 and $\frac{1}{\cos\theta}$. As θ is diminished, $\cos\theta$ approaches 1; and the smaller θ becomes, the more nearly does $\frac{1}{\cos\theta}$ approach 1. [Art. 94]

Therefore, by diminishing θ sufficiently, we can make $\frac{\theta}{\sin\theta}$ differ from 1 by less than any assignable quantity however small.

This is what is meant when it is said that '*the limit of* $\frac{\theta}{\sin\theta}$, *when* $\theta = 0$, *is* 1.'

291. The student must notice carefully that θ here is the **number of radians** in the angle referred to.*

Example. Prove that the limit of $\frac{n}{\sin n^0}$, *when* $n = 0$, *is* $\frac{180}{\pi}$.

Let θ radians $= n^0$, then

$$\frac{\theta}{\pi} = \frac{n}{180}, \quad \therefore n = \frac{180\theta}{\pi}, \quad \therefore \frac{n}{\sin n^0} = \frac{180}{\pi} \cdot \frac{\theta}{\sin\theta}.$$

When n is diminished, θ is diminished also, and the limit of $\frac{\theta}{\sin\theta}$, when $\theta = 0$, is 1. Therefore the limit of $\frac{n}{\sin n^0}$ $\left(\text{which } = \frac{180}{\pi} \cdot \frac{\theta}{\sin\theta}\right)$, when $n = 0$, is $\frac{180}{\pi}$.

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* It is on this account that a radian is used as a unit of angle (Art. 62).

292. *To prove that if* θ *be the circular measure of a positive angle less than a right angle,* $\sin\theta$ *lies between* θ *and* $\theta - \frac{1}{4}\theta^3$.

It has been proved that $\sin\theta$ is less than θ .... (i).

And that θ is less than $\tan\theta$.

Therefore $\frac{\theta}{2}$ is less than $\tan\frac{\theta}{2}$; or, $\frac{\theta}{2}\cos\frac{\theta}{2}$ is less than $\sin\frac{\theta}{2}$.

Now, $\sin\theta = 2\sin\frac{\theta}{2}\,.\,\cos\frac{\theta}{2}$; [Art. 166]

$$\therefore \sin\theta \text{ is greater than } 2\left(\frac{\theta}{2}\cos\frac{\theta}{2}\right)\cos\frac{\theta}{2},$$

i.e. greater than $\theta\cos^2\frac{\theta}{2}$, i.e. greater than $\theta - \theta\sin^2\frac{\theta}{2}$.

But $\left(\frac{\theta}{2}\right)^2$ is greater than $\sin^2\frac{\theta}{2}$.

Still greater therefore is $\sin\theta$ than $\theta - \frac{1}{4}\theta^3$.........(ii).

Hence, $\sin\theta$ is less than θ, and greater than $\theta - \frac{\theta^3}{4}$.

293. *To find* sin 10″.

In the above, θ is the *circular measure* of the angle.

The circular measure of 10″, correct to three significant figures, is ·0000484.... [Examples X. (17).]

Let $\theta = \cdot0000484....$

Then,

$$\theta - \tfrac{1}{4}\theta^3 = \theta - \tfrac{1}{4}(\cdot0000484....)^3 = \theta - \cdot000000000000028....$$

Hence, θ and $(\theta - \frac{1}{4}\theta^3)$ are decimal fractions which agree in their first twelve figures at least.

And since $\sin 10''$ lies *between* these fractions, therefore the first twelve decimal places of $\sin 10''$ are the same as those of the circular measure of $10''$.

Hence, if the value of π be given to a sufficient number of decimal places, we can calculate the circular measure of $10''$, and therefore also $\sin 10''$, to 12 decimal places.

294. *To show how to construct a Table giving the Trigonometrical Ratios of angles which form an arithmetical progression having* $10''$ *for the common difference.*

In the identity

$$\sin(n+1)\alpha + \sin(n-1)\alpha = 2\sin n\alpha \,.\, \cos\alpha,$$

let $\alpha = 10''$, and suppose a table of the sines of all angles at intervals of $10''$ to have been calculated up to $n \,.\, 10''$.

Then, $\sin(n-1)\alpha$, $\sin n\alpha$ and $\cos\alpha \left[= \sqrt{1-\sin^2 10''}\right]$ are known.

Therefore by the above formula $\sin(n+1)\alpha$ can be found.

Hence, since we know $\sin 10''$, the sine of $20''$ can be found; and then the value of $\sin 30''$; and so on.

295. When the sines of angles up to $45°$ have been calculated, the rest may be found from the formula,

$$\sin(45° + A) - \sin(45° - A) = \sqrt{2} \,.\, \sin A.$$

Also, when the sines of angles up to $60°$ have been calculated, the remainder up to $90°$ can be found still more easily from the formula, $\sin(60° + A) - \sin(60° - A) = \sin A$.

The other ratios may be found from the following:

$$\cos A = \sin(90° - A), \quad \tan A = \frac{\sin A}{\cos A}, \quad \cot A = \tan(90° - A),$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \quad \sec A = \operatorname{cosec}(90° - A).$$

296. Since $3^\circ = 18^\circ - 15^\circ$, the sines of angles differing by 3°, or by any multiple of 3°, can be found independently. (See p. 120.) These values may be used to *test* the accuracy of the Tables calculated as above.

297. The following formulæ may be also used to test the accuracy of the Tables,

$$\sin(36^\circ+A)-\sin(36^\circ-A)-\sin(72^\circ+A)+\sin(72^\circ-A)=\sin A,$$

$$\cos(36^\circ+A)+\cos(36^\circ-A)-\cos(72^\circ+A)-\cos(72^\circ-A)=\cos A.$$

They are called **formulæ of verification.**

ON THE LIMIT OF THE VISIBLE HORIZON.

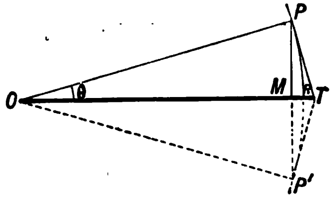
298. The surface of the sea is very nearly that of a sphere whose radius is 3957 miles.

The height of the highest mountains on the globe is less than 6 miles. Thus a point must be considered to be at a very considerable height above the surface of the sea if its height is a thousandth part of the earth's radius.

299. In the figure, let O be the centre of the earth, PRP' part of the surface of the sea, T a point of observation, TR its vertical height.

Draw TP, TP' tangents to the earth's surface. Then PMP' is parallel to the 'horizontal plane' at R.

The angle TPM is called the **dip of the horizon** at T. It is the angle of depression of the most distant point on the horizon seen from T. It must obviously be a very small angle, since TR is so small compared with RO.



300. If RO be produced to cut the circle again in L,

then $$TP^2 = TR \,.\, TL \qquad \text{[Euc. III. 36]}$$
$$= TR\,(RL + TR) = 2TR \,.\, RO + TR^2.$$

But TR will in general be much less than a thousandth part of RO, and therefore TR^2 will be much less than a thousandth part of $2TR \,.\, RO$.

Hence, the formula $TP^2 = 2TR \,.\, RO$, i.e. $TP^2 =$ twice the earth's radius × vertical height, will give the value of TP correct to at least three significant figures.

Example. Three times the height in feet *of the place of observation above the sea is equal to twice the square of the distance of the horizon* in miles.

Here, $TP^2 = RL \times 7914$ miles.

Let f be the number of feet in RL, then the number of miles in RL is $\frac{f}{5280}$; let x be the number of miles in TP, then

$$x^2 = \frac{f}{5280} \times 7914 = \frac{3f}{2} \text{ nearly.} \quad \text{Q.E.D.}$$

EXAMPLES. LXXV.

(1) Show that the limit of $\frac{n}{2} R^2 \sin \frac{2\pi}{n}$ (i.e. the area of a polygon of n sides inscribed in a circle of radius R), when $n = \infty$ is πR^2.

(2) Prove that the limit of $nr^2 \tan \frac{\pi}{n}$, when $n = \infty$, is πr^2.

(3) Given that $\pi = 3{\cdot}141592653589793\ldots$ prove that the circular measure of $10''$ is $\cdot00004848136811\ldots$

(4) Prove that $2 \sin (72^\circ + A) - 2 \sin (72^\circ - A) = (\sqrt{5} - 1) \sin A$, and that $2 \sin (36^\circ + A) - 2 \sin (36^\circ - A) = (\sqrt{5} + 1) \sin A$.

(5) If a mast of a ship be 150 feet high, show that the greatest distance seen from its top is 15 miles nearly.

(6) Prove that if the dip of the horizon at the top of a mountain is $1^\circ 26'$ [$= \tan^{-1} \cdot025$], the mountain is about 6530 feet high.

NOTE. The definitions given in Arts. 75, 78 of the Trigonometrical Ratios are now used exclusively.

The NAMES *tangent*, *secant*, *sine*, were given originally to quantities defined as follows.

Let *ROP* be any angle. With centre *O* and any radius describe the arc *RP*. Draw *PM* perpendicular to *OR* and *PT* perpendicular to *OP*. (See Figure on previous page.)

Then *PR* is called an *arc*, *PT* is the *tangent of the arc PR*, *OT* is the *secant of the arc PR*, *MP* is the *sine of the arc PR*.

The name *sine* is derived from the word *sinus*. For, in the figure, *PMP′* is the string of the "bow" (*arcus*), and the string of a bow when in use is pulled to the archer's *breast*.

The *co*-tangent, *co*-secant and *co*-sine are respectively the tangent, secant and sine of the *complement* of the arc or of the angle.

The sine, tangent, etc. of the *angle* are the same as the *measures* of the sine, tangent, etc. of the *arc*, when the radius of the circle is the unit of length.

**MISCELLANEOUS EXAMPLES. LXXVI.

(1) If $2\cos\theta - \cos 2\theta = a$ and $2\sin\theta - \sin 2\theta = b$, prove that $(a^2+b^2-3)^2 = 12-8a$.

(2) If $h\cos\theta + k\sin\theta = 1$ and $l\cos\theta + m\sin\theta = 1$, prove that $(l-h)^2+(m-k)^2 = (lh-mk)^2$.

(3) The diagonals of a rhombus are $2a$ and $2b$, prove that the cosines of its angles are $\pm\dfrac{a^2\sim b^2}{a^2+b^2}$.

(4) One of the values of

$$\sqrt{}\,[2+\sqrt{}\,\{2+\sqrt{}\,(2+\ldots+\sqrt{2+2\cos A})\}] \text{ is } 2\cos\frac{A}{2^n}.$$

(5) If $8x = \log_e 3$, prove that the angle whose tangent is $\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ is 15^0.

(6) If the number of degrees (A) in an angle is the same as the number of radians in another angle, and if the tangents of these angles are equal, prove that A is some multiple of $\dfrac{180\pi}{180-\pi}$.

(7) If α, β, γ are in A.P., then

$$\sin\alpha + \sin\gamma = 2\sin\beta \, . \cos(\beta-\alpha).$$

(8) Two parallel chords of a circle, lying on the same side of the centre, subtend respectively 72^0 and 144^0 at the centre. Prove that the distance between the chords is half the radius of the circle.

(9) Prove that $4\sin(\theta-\alpha) \, . \sin(m\theta-\alpha) \, . \cos(\theta-m\theta)$

$$= 1+\cos(2\theta-2m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).$$

(10) Express $x^4+y^4+z^4-2y^2z^2-2z^2x^2-2x^2y^2$ in a form fitted for logarithmic calculation.

(11) Solve the equations

(i) $\sin 3\theta + \sqrt{3}\cos 3\theta = 1$. (ii) $\sin m\theta = \cos n\theta$.

(iii) $\dfrac{\cos(\beta + x)}{\cos(\alpha - x)} = \dfrac{m\sin\beta}{n\sin\alpha}$. (iv) $\tan m\theta = \cot n\theta$.

(v) $\tan\theta + \tan 2\theta + \tan 3\theta = 0$.

(vi) $\cos 8\theta - \cos 5\theta + \cos 3\theta = 1$.

(vii) $\cos\theta . \cos 3\theta = \cos 5\theta . \cos 7\theta$.

(12) In any triangle $\tan\dfrac{A-B}{2} = \tan\left(\phi - \dfrac{\pi}{4}\right)\cot\dfrac{C}{2}$, where $\phi = \tan^{-1}\dfrac{a}{b}$.

(13) Prove that

$$b^2 + c^2 - 2bc\cos(60^0 + A) = c^2 + a^2 - 2ca\cos(60^0 + B).$$

Hence prove that if equilateral triangles are described on the sides of the triangle ABC, the centres of circles inscribed in them will be the angular points of an equilateral triangle.

(14) A round tower stands on an island in a lake. A, B are two points on the land such that AB is a feet and points directly to the middle of the tower. At A and B the base of the tower subtends the angles 2α and 2β respectively. Prove that the diameter of the tower is $2\dfrac{a\sin\alpha . \sin\beta .}{\sin\beta - \sin\alpha}$.

(15) At P the top of a tower of height h, the angles of depression of two objects on the horizontal plane on which the tower stands are $\frac{1}{4}\pi - \alpha$ and $\frac{1}{4}\pi + \alpha$. Prove that the angle $APB = 2\alpha$, and that $AB = 2h\tan 2\alpha$.

(16) On the side of a hill there are two places B, C inaccessible to each other, but known to be at the same distance (a) from a certain station A also on the hill. At the lower place C the *horizontal* angle θ between A and B is observed as well as the altitudes λ, μ of A and B. Prove that the distance between B and C is $2a\left\{\cos(\lambda - \mu)\cos^2\dfrac{\theta}{2} - \cos(\lambda + \mu)\sin^2\dfrac{\theta}{2}\right\}$.

(17) In the ambiguous case, given B, c, b, if a_1, a_2 are the two values of a, prove the following statements:

(i) $a_1 + a_2 = 2c \cos B$. (ii) $a_1 a_2 = c^2 - b^2$.

(iii) $a_1^2 - 2a_1 a_2 \cos 2B + a_2^2 = 4b^2 \cos^2 B$.

(iv) The distance between the centres of the circumscribing circles of the two triangles is $\dfrac{a_1 - a_2}{2 \sin B}$.

(v) The circumscribing circles of the two triangles are equal.

(vi) If $B = 45^\circ$, the angle between the two positions of l is

$$\cos^{-1} \frac{2a_1 a_2}{a_1^2 + a_2^2}.$$

(18) Prove that if I_1, I_2, I_3 are the centres of the escribed circles, then $I_2 A I_3$ is a straight line, and $I_1 A$, $I_2 B$, $I_3 C$ are perpendicular to $I_2 I_3$, $I_3 I_1$, $I_1 I_2$ respectively. Prove also that

(i) $I_2 I_3 = a \operatorname{cosec} \dfrac{A}{2}$. (ii) The angle $I_2 I_1 I_3 = \dfrac{\pi}{2} - \dfrac{A}{2}$.

(iii) The area of $I_1 I_2 I_3 = \dfrac{sa}{\sin A} = \dfrac{\frac{1}{2} S}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}$.

(iv) The radius of the circle circumscribing $I_1 I_2 I_3 = 2R$.

(19) If D, E, F are the feet of the perpendiculars from A, B, C on the opposite sides, the triangle DEF is called the 'pedal triangle' of the triangle ABC*; prove that

(i) $EF = a \cos A$. (ii) The angle $EDF = \pi - 2A$.

(iii) $\dfrac{1}{r} = \dfrac{1}{AD} + \dfrac{1}{BE} + \dfrac{1}{CF}$. (iv) $\dfrac{AD^2}{BE \cdot CF} = \dfrac{bc}{a^2}$.

(v) The radius of the circle circumscribing $AEF = R \cos A$.

(vi) The radius of the circle circumscribing $DEF = \frac{1}{2} R$.

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* The student should notice that ABC is the pedal triangle of $I_1 I_2 I_3$ in Question (18).

(20) Prove that in any triangle ABC

$$a^2 = (b-c)^2\left\{1 + \frac{4bc}{(b-c)^2}\sin^2\frac{A}{2}\right\}.$$

Hence prove that if $\tan\theta = \frac{2\sqrt{bc}}{b-c}\sin\frac{A}{2}$, then $a = (b-c)\sec\theta$.

(21) Prove that in any triangle ABC

$$a = (b+c)\cos\beta, \text{ where } \sin\beta = \frac{2\sqrt{bc}}{b+c}\cos\frac{A}{2}.$$

Use this method, or that of Example (20), to find a by the aid of the Tables, when $b = 347$, $c = 293$, $A = 39^0\ 42'$. [See Art. 260.]

(22) Prove that in any triangle

$$\log a = \log(b-c) + L\cos\frac{A}{2} - L\cos\phi\ ; \text{ where } \phi = \tan^{-1}\frac{b+c}{b-c}\tan\frac{A}{2}.$$

(23) If D, E, F are the middle points of BC, CA, AB, then

$$\sin BAD = \frac{b\sin A}{\sqrt{b^2 + 2bc\cos A + c^2}}.$$ [See Ex (11) (15) page 205.]

(24) If $x = II_1$, $y = II_2$, $z = II_3$, $d = 2R$, prove that

$$xyz + d(x^2 + y^2 + z^2) = 4d^3.$$

(25) The circle escribed to the side BC of the triangle ABC will be the inscribed circle of the triangle whose sides are

$$\frac{sa}{s-a},\ \frac{sb}{s-a},\ \frac{sc}{s-a}.$$

(26) If a point Q be taken within ABC such that the angles BQC, CQA, AQB are each 120^0, prove that

$$OA = \tfrac{1}{2}\cdot\frac{4\sqrt{2}\Delta + \sqrt{6}(b^2 + c^2 - a^2)}{\{12\sqrt{3}\Delta + 3(a^2 + b^2 + c^2)\}^{\frac{1}{2}}}.$$

(27) If $\sin A$ and $\cos A$ are both given, prove that in general n different values and no more can be assigned to $\sin\frac{A}{n}$.

(28) If from a point on a circular arc (of radius r) which subtends an angle a at the centre, perpendiculars are drawn to its bounding radii, then the distance between the feet of the perpendiculars is $r \sin a$.

(29) If two circles, radii a and b, touch externally, the angle between their common tangents $=2\tan^{-1}\frac{1}{2}\left(\sqrt{\frac{a}{b}} \sim \sqrt{\frac{b}{a}}\right)$.

(30) If the inscribed circle touches the sides in DEF, prove that $EF : FD : DE = \cos\frac{A}{2} : \cos\frac{B}{2} : \cos\frac{C}{2}$.

(31) If the bisectors of the angles A, B, C meet the opposite sides in D, E, F then (i) $AD = \frac{2bc}{b+c}\cos\frac{A}{2}$. (ii) $CD = \frac{a\sin B}{\sin B + \sin C}$.

(iii) The area of DEF

$$= S\frac{2abc}{(b+c)(c+a)(a+b)} = S \cdot \frac{2\sin\frac{A}{2} \cdot \sin\frac{B}{2} \cdot \sin\frac{C}{2}}{\cos\frac{B-C}{2} \cdot \cos\frac{C-A}{2} \cdot \cos\frac{A-B}{2}}.$$

(32) If $\alpha, \beta, \gamma, \delta$ are the angles of a quadrilateral inscribed in a circle, then

$$\cos(\alpha+\beta) \cdot \cos(\beta+\gamma) \cdot \cos(\gamma+\delta) \cdot \cos(\delta+\alpha) = (1 - \cos^2\alpha - \cos^2\beta)^2.$$

(33) If θ_1 and θ_2 be two values of θ found from the equation $a\cos\theta + b\sin\theta = c$, then

$$\frac{1}{a} \cdot \cos\frac{\theta_1+\theta_2}{2} = \frac{1}{b} \cdot \sin\frac{\theta_1+\theta_2}{2} = \frac{1}{c} \cdot \cos\frac{\theta_1-\theta_2}{2}.$$

(34) If $\left.\begin{aligned} a &= b\cos\chi + c\cos\phi \\ b &= c\cos\theta + a\cos\chi \\ c &= a\cos\phi + b\cos\theta \end{aligned}\right\}$

prove that $\cos 2\theta + \cos 2\phi + \cos 2\chi + 4\cos\theta \cdot \cos\phi \cdot \cos\chi + 1 = 0$,

hence [see Ex. (20), p. 143], prove that

$$(\theta \pm \phi \pm \chi) = (2n+1)\pi.$$

(35) A circle is drawn touching the inscribed circle and the sides *AB*, *AC* (not produced) of the triangle *ABC*. Another circle is drawn touching the circle and the same side. Prove that the radius of the n^{th} circle thus drawn is $r\left(\dfrac{1-\sin\dfrac{A}{2}}{1+\sin\dfrac{A}{2}}\right)^{n}$ and that the sum of the radii of all the circles, when $n=\infty$, is

$$\frac{r}{2}\left(\operatorname{cosec}\frac{A}{2}-1\right).$$

(36) If *AO*, *BO*, *CO* pass through the centre of the circumscribing circle and meet the opposite sides on *DEF*, then

$$\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}=\frac{2}{R}.$$

(37) If h and k be the lengths of the diagonals of a quadrilateral and θ the angle between them, prove that the area of the quadrilateral is $\frac{1}{2}hk\sin\theta$.

(38) If $\dfrac{x}{\sin X}=\dfrac{y}{\sin Y}=\dfrac{z}{\sin Z}$ and $X+Y+Z=180^\circ$, prove that $x=y\cos Z+z\cos Y$.

(39) Two regular polygons of n and $2n$ sides are described such that the circle inscribed in the first circumscribes the second; also the radius of the circle inscribed in the second is to that of the circle circumscribing the first as $3+\sqrt{3}$ is to $4\sqrt{2}$. Prove that $n=6$.

(40) Prove by the aid of a figure like that of Art. 282 that $I_1O^2=R^2+2Rr_1$.

(41) If $yz+zx+xy=1$, prove by Trigonometry that

$$\frac{x}{1-x^2}+\frac{y}{1-y^2}+\frac{z}{1-z^2}=\frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

[See Ex. (33), (32), p. 194.]

ANSWERS TO THE EXAMPLES.

I. Page 2.

(1) 80. (2) 10. (3) 16. (4) $109\frac{11}{21}$.

(5) 5 acres. (6) $\frac{1760a}{b}$. (7) $\frac{a \, . \, c}{3}$ yds.

(8) A shilling and a three-penny piece.

II. Pages 4, 5.

(1) 77440. (2) 6 ft. (3) $\frac{125}{1}$. (4) $6\frac{3}{4}$, $\frac{27}{4}$, $\frac{27}{4}$.

(5) 15·625, $15\frac{5}{8}$, $\frac{125}{8}$, $\frac{125}{8}$. (6) $9\frac{9}{10}$, 9 shillings.

(7) $\frac{80a}{b}$, $\frac{80a}{c}$, $\frac{80a}{d}$, $\frac{80a}{k}$. (8) 21 shillings.

III. Pages 7, 8.

(1) 10 ft. (2) 80 yds. (3) 20 ft. (4) 50 ft.

(5) 90 ft. (6) $20\frac{10}{27}$ nearly. (7) $5a$ feet.

(8) $12a$ yards. (10) $\frac{\sqrt{2}}{2}\,a$ yards. (12) $\frac{2\sqrt{3}}{3}\,a$ feet.

(13) $1 : \sqrt{2}$. (14) $\sqrt{84}$ ft. (15) $2\sqrt{9a^2 - b^2}$ ft.

IV. Page 13.

(1) 9·899 ft. (2) 2489 yds. (3) 5836 in.

(4) 1138 in. (5) 933·4 ft. (6) 2504 in.

(7) 8·66 ft. (8) 4·607 ft. (9) 48 in.

(10) 85 ft. 11 in.

V. Pages 21, 22.

(1) $3\frac{1}{7}$ yds. (2) $25\frac{1}{7}$ ft. (3) $150\frac{6}{7}$ in. (4) $3\frac{2}{11}$ ft.

(5) $7\frac{7}{11}$ ft. (6) 560. (7) $15\frac{1}{2}$ nearly. (8) 33600.

(9) $32\frac{1}{7}$. (10) 7 ft. (11) $553\frac{1}{7}$, 13·8 in. (12) $339\frac{3}{7}$ ft.

(13) 443 in. (14) 235 in. (15) 203 in.

(16) 274 in. (17) 1886 in.

VII. Page 28.

(1) ·632118 of a right angle. (3) ·021827 of a right angle.

(2) 1·0426991 ,, (4) ·03294894 ,,

(5) ·006241 of a right angle. (9) ·6900071 of a right angle.
(6) 10·000812 ,, (10) 1·19030045 ,,
(7) ·3204052 ,, (11) 10·061801 ,,
(8) ·0102034 ,, (12) ·0226048 ,,

(13) $36^{g}\,78^{`}\,91^{``}$. (14) $104^{g}\,30^{`}\,21^{``}$. (15) $1^{g}\,20^{`}\,3^{``}$.
(16) $10^{`}\,20^{``}$. (17) $6^{g}\,25^{`}$. (18) $302^{g}\,12^{`}\,50^{``}$.
(19) $100^{g}\,10^{`}$. (20) $1^{g}\,1^{`}\,\cdot1^{``}$. (21) $645^{g}\,10^{`}$.
(22) $2^{g}\,30^{`}$. (23) $1^{`}\,10^{``}$. (24) $10^{``}$.

VIII. Page 30.

(1) ·09175 of a right angle $=9^{g}\,17^{`}\,50^{``}$.
(2) ·0675 ,, $=6^{g}\,75^{`}$.
(3) 1·07875 ,, $=107^{g}\,87^{`}\,50^{``}$.
(4) $\cdot18042\dot{9}01234567\dot{9}$ $=18^{g}\,4^{`}\,29^{``}$, etc.
(5) $1\cdot46\dot{7}$,, $=146^{g}\,77^{`}\,77\cdot\dot{7}^{``}$.
(6) $\cdot5\dot{4}$,, $=54^{g}\,44^{`}\,44\cdot\dot{4}^{``}$.
(7) $1^\circ\,14'\,15''$. (8) $7^\circ\,52'\,30''$. (9) $153^\circ\,24'\,29\cdot34''$.
(10) $21^\circ\,36'\,8\cdot1''$. (11) $16^\circ\,12'\,37\cdot26''$. (12) $31^\circ\,30'$.

IX. Page 35.

1. (1) 2 right angles or 180°. (2) $\frac{2}{3}$ of a right angle.
(3) $\frac{2}{\pi}$ right angles. (4) $\frac{6}{\pi}$ right angles.
(5) 2 right angles. (6) $\frac{4}{\pi^2}$ right angles.
(7) $\frac{2\theta}{\pi}$ right angles. (8) ·002 of a right angle.
(9) 20 right angles.

2. (1) π. (2) 2π. (3) $\frac{\pi}{3}$. (4) $\frac{\pi}{8}$.
(5) $\frac{\pi}{180}$. (6) 1^c. (7) $\frac{n}{180}\pi$. (8) $\frac{1}{2}^c$. (9) $\frac{A\pi}{180}$.

3. (1) $\frac{\pi}{6}$. (2) $\frac{\pi}{4}$. (3) $\frac{\pi}{12}$. (4) $\frac{\pi}{200}$.
(5) $\frac{\pi}{20000}$. (6) $\frac{\pi}{200000}$. (7) $\frac{n\pi}{200}$. (8) 1^c. (9) 5π.

4. (1) $\frac{1}{3}$. (2) $\frac{10}{9}$. (3) 1. (4) $\frac{50}{3\pi}$. (5) $\frac{33}{20}$. (6) $\frac{\pi}{180}$.

X. Pages 37—39.

(1) $\frac{3}{2}$. (2) 90. (3) $4\frac{4}{5}$. (4) $\frac{700}{11}$.
(5) $112\frac{1}{2}$ ft. (6) $5\frac{37}{88}$ ft. (7) $9\frac{3}{7}$ ft.
(8) 838000 miles. (9) $\frac{1}{9}$ radian $=6\frac{4}{11}$ degrees.
(10) $21\frac{21}{44}$ degrees. (11) $51\frac{6}{11}''$. (12) about 34 yds.
(13) 1 : 3·1416. (14) 3·1416. (15) $3\cdot141\dot{6}$.
(16) 400 : 1. (17) ·0000484.... (18) $49\frac{1}{11}$ in.
(19) $\frac{\pi}{2}$ i.e. a right angle. (21) 473 : 489.
(22) (i) $k=1$, (ii) $k=\frac{180}{\pi}$.

XI. Pages 40—43.

(1) $\frac{\pi}{10}$. (2) 33°, 15°. (3) 90°, 45°. (4) 18°, $22\frac{1}{2}^\circ$.
(5) 38°, 18°. (7) 45°, 60°, 75°. (8) 39°, 60°, 81°.
(9) 30°, 60°, 90°. (14) 25°, $8^\circ\,6'$. (15) $\frac{11}{15}$.
(16) $\frac{n\pi}{10800}$. (17) (i) 120°, $133\cdot\dot{3}^g$, $\frac{2\pi}{3}$, (ii) 135°, 150^g, $\frac{3\pi}{4}$,
(iii) 156°, $173\cdot\dot{3}^g$, $\frac{13\pi}{15}$. (20) $450'$. (21) (i) $3\frac{3}{4}$, (ii) $\frac{15}{2\pi}$.
(22) $\frac{13}{17}^\circ$. (23) a right angle. (24) $\frac{10ac}{9b}$.
(25) $\frac{9a+10b}{10c}$ degrees. (26) $\frac{5\pi}{9}$. (27) $\frac{5\pi}{19}$, $\frac{9\pi}{38}$, $\frac{\pi}{2}$.
(28) $\frac{1800\pi}{19\pi+1800}$. (29) $\frac{\pi^2}{10\pi+18000}$. (30) 9 or 16.

XII. Pages 50—52.

(1) (i) DA, BD. (ii) DB, AD. (iii) DA, CD. (iv) DC, AD.
(2) (i) $\frac{DB}{AB}$. (ii) $\frac{DC}{CA}$. (iii) $\frac{CD}{AD}$. (iv) $\frac{DA}{BA}$. (v) $\frac{DB}{AD}$.
(vi) $\frac{DC}{AC}$. (vii) $\frac{CD}{CA}$. (viii) $\frac{DA}{CD}$. (ix) $\frac{BD}{BA}$. (x) $\frac{DA}{CA}$.
(3) (i) $\frac{DB}{CB}$, $\frac{BA}{CA}$. (ii) $\frac{CD}{CB}$, $\frac{CB}{CA}$. (iii) $\frac{DB}{CD}$, $\frac{BA}{CB}$.
(iv) $\frac{DB}{AB}$, $\frac{BC}{AC}$. (v) $\frac{AD}{AB}$, $\frac{AB}{AC}$. (vi) $\frac{DB}{AD}$, $\frac{BC}{AB}$.

(4) (i) $\frac{DA}{BA}$. (ii) $\frac{BA}{EA}$ or $\frac{AC}{EC}$. (iii) $\frac{DC}{BC}$. (iv) $\frac{AB}{AE}$.

(v) $\frac{AD}{AB}$ or $\frac{AB}{AC}$. (vi) $\frac{BD}{BC}$. (vii) $\frac{DB}{CD}$, or $\frac{BA}{CB}$, or $\frac{AE}{CA}$.

(viii) $\frac{DA}{BD}$. (ix) $\frac{BA}{EB}$ or $\frac{AC}{EA}$. (x) $\frac{DC}{BD}$. (xi) $\frac{DB}{AB}$ or $\frac{BC}{AC}$.

(xii) $\frac{BE}{AE}$.

(5) $\sin A = \frac{3}{5}$, $\cos A = \frac{4}{5}$, $\tan A = \frac{3}{4}$; $\sin B = \frac{4}{5}$, $\cos B = \frac{3}{5}$, $\tan B = \frac{4}{3}$.

(7) Of the smaller angle, the sine $= \frac{5}{13}$, cosine $= \frac{12}{13}$, tangent $= \frac{5}{12}$.
Of the larger angle, the sine $= \frac{12}{13}$, cosine $= \frac{5}{13}$, tangent $= \frac{12}{5}$.

(8) Of the smaller angle, the sine $= \frac{1}{2}$, cosine $= \frac{\sqrt{3}}{2}$, tangent $= \frac{1}{\sqrt{3}}$.

Of the larger angle, the sine $= \frac{\sqrt{3}}{2}$, cosine $= \frac{1}{2}$, tangent $= \sqrt{3}$.

XIV. Pages 63—65.

(1) 179 ft. (2) 346 ft. (3) 86·6 ft. (4) 138·5 ft.
(5) $7\frac{1}{2}$ ft. (6) 60°, 173 ft. (7) 63·17 yds. (8) 34·15 ft.
(9) 73·2 ft. (10) 86·6 ft. (11) ·866 miles = 1524 yds.
(12) 173·2 yds. (14) 373 ft. (15) 3733 ft.
(16) $\frac{\sqrt{6}}{2}$ miles = 6465 ft. (17) $\frac{\sqrt{3}\,.\,a}{3b}$. (18) 30°.
(19) about 523·6 miles.

XVI. Page 74.

(1) $\sin A = \sqrt{1-\cos^2 A}$, $\tan A = \frac{\sqrt{1-\cos^2 A}}{\cos A}$, $\cot A = \frac{\cos A}{\sqrt{1-\cos^2 A}}$,

$\sec A = \frac{1}{\cos A}$, $\operatorname{cosec} A = \frac{1}{\sqrt{1-\cos^2 A}}$.

(2) $\sin A = \frac{1}{\sqrt{1+\cot^2 A}}$, $\cos A = \frac{\cot A}{\sqrt{1+\cot^2 A}}$, $\tan A = \frac{1}{\cot A}$,

$\sec A = \frac{\sqrt{1+\cot^2 A}}{\cot A}$, $\operatorname{cosec} A = \sqrt{1+\cot^2 A}$.

(3) $\sin A = \frac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\cos A = \frac{1}{\sec A}$, $\tan A = \sqrt{\sec^2 A - 1}$,

$\cot A = \frac{1}{\sqrt{\sec^2 A - 1}}$, $\operatorname{cosec} A = \frac{\sec A}{\sqrt{\sec^2 A - 1}}$

(4) $\sin A = \dfrac{1}{\operatorname{cosec} A}$, $\cos A = \dfrac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$, $\tan A = \dfrac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$,

$\cot A = \sqrt{\operatorname{cosec}^2 A - 1}$, $\sec A = \dfrac{\operatorname{cosec} A}{\sqrt{\operatorname{cosec}^2 A - 1}}$.

(5) $\cos A = \sqrt{1 - \sin^2 A}$, $\tan A = \dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$, $\cot A = \dfrac{\sqrt{1 - \sin^2 A}}{\sin A}$,

$\sec A = \dfrac{1}{\sqrt{1 - \sin^2 A}}$, $\operatorname{cosec} A = \dfrac{1}{\sin A}$.

(6) $\sin A = \dfrac{\tan A}{\sqrt{1 + \tan^2 A}}$, $\cos A = \dfrac{1}{\sqrt{1 + \tan^2 A}}$, $\cot A = \dfrac{1}{\tan A}$,

$\sec A = \sqrt{1 + \tan^2 A}$, $\operatorname{cosec} A = \dfrac{\sqrt{1 + \tan^2 A}}{\tan A}$.

XVII. Page 75.

(1) $\frac{3}{4}$, $\frac{4}{5}$. (2) $\dfrac{2\sqrt{2}}{3}$, $\dfrac{1}{2\sqrt{2}}$. (3) $\frac{4}{3}$, $\frac{5}{3}$.

(4) $\dfrac{1}{\sqrt{15}}$, $\dfrac{\sqrt{15}}{4}$. (5) $\dfrac{\sqrt{3}}{2}$, $\frac{1}{2}$. (6) $\dfrac{\sqrt{5}}{3}$, $\frac{2}{3}$.

(7) $\dfrac{b}{\sqrt{c^2 - b^2}}$. (8) $\dfrac{a}{\sqrt{a^2 + b^2}}$, $\dfrac{b}{\sqrt{a^2 + b^2}}$.

(9) $\dfrac{\sqrt{a^2 - 1}}{a}$, $\dfrac{1}{\sqrt{a^2 - 1}}$. (11) $h^2(1 + k^2) = 1$.

XVIII. Page 77.

(2) $\sec\theta$ increases continuously from 1 to ∞.

(3) $\sin A$ diminishes continuously from 1 to 0.

(4) $\cot\theta$ diminishes continuously from ∞ to 0.

XIX. Page 82.

(4) Yes. (5) No. (6) Yes.

(7) (i) 60°. (ii) -100°. (iii) 0°. (iv) -260°. (v) 115°.

(vi) 410°. (vii) $-\dfrac{\pi}{4}$. (viii) $\dfrac{2\pi}{3}$.

XX. Page 84.

(1) 45°. (2) 30°. (3) 45°. (4) 60°.

(5) 30°. (6) 30°. (7) 30°. (8) 0°, or 45°.

(9) 90°, or 60°. (10) 60°. (11) 45°. (12) 45°.
(13) 90°, or 45°. (14) 18°. (15) 45°. (16) 45°.
(17) 30°. (18) 30°.

XXI. Page 85.

(3) the value 3 is inadmissible. (4) $\frac{2 \pm \sqrt{2}}{4}$.

(5) $\frac{3}{4}$, or $\frac{1}{2}$. (6) $\frac{3}{4}$, or $\frac{1}{3}$.

(7) the value $-\frac{7\sqrt{3}}{6}$ is inadmissible. (9) $1 - \sin^4\theta$.

(10) $1 - 3\sin^2\theta + 3\sin^4\theta$. (11) $\frac{1 - 2\cos^2\theta + 2\cos^4\theta}{\cos^4\theta}$.

(13) $\frac{1 - \sin A}{1 + \sin A}$.

(14) cosec θ decreases continuously from ∞ to 1.

(15) cot θ increases continuously from 0 to ∞.

(16) $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{12}$.

XXII. Page 89.

(1) +6. (2) 0. (3) +2. (4) +3.
(5) +10. (6) 0. (7) +7. (8) +7.

XXIV. Page 94.

(1) The second. (2) The fourth. (3) The second.
(4) The third. (5) The fourth. (6) The first.
(7) The second. (8) The first. (9) The first.
(10) The fourth. (11) The fourth.
(12) The first, if n be even, the third, if n be odd.

XXV. Page 98.

(1) +, +, +. (2) +, −, −. (3) −, −, +.
(4) −, +, −. (5) −, +, −. (6) −, −, +.
(7) +, −, −. (8) +, +, +. (9) +, +, +.
(10) +, +, +. (11) +, −, −. (12) −, +, −.

XXVI. Page 100.

(1) $+\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$. (2) $+\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1$.

(3) $+\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}$. (4) $-\frac{1}{2}, +\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}$.

(5) $-\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, -1.$ (6) $+\frac{\sqrt{3}}{2}, +\frac{1}{2}, +\sqrt{3}.$

(7) $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1.$ (8) $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, +1.$

(9) $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$ (10) $+\frac{1}{2}, +\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$

(11) $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, +\sqrt{3}.$ (12) $-\frac{\sqrt{3}}{2}, +\frac{1}{2}, -\sqrt{3}.$

(13) $+\frac{1}{\sqrt{2}}, +\frac{1}{\sqrt{2}}, +1.$ (14) $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$

(15) $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, +\frac{1}{\sqrt{3}}.$

XXVII. Page 100.

Each of these expressions changes *continuously* as the angle A increases from 0^0 to 360^0, and their values are *repeated* at each complete revolution of the revolving line.

Their values are given below when they are zero and when they cease to increase and begin to decrease, and vice versa. The first table gives also the sign of each between the values.

	0^0		90^0		180^0		270^0	
(1) $\cos A$	+1	+	0	−	−1	−	0	+
(2) $\tan A$	0	+	∞	−	0	+	∞	−
(3) $\cot A$	∞	+	0	−	∞	+	0	−
(4) $\sec A$	+1	+	∞	−	−1	−	∞	+
(5) $\operatorname{cosec} A$	∞	+	+1	+	∞	−	−1	−
(6) $1-\sin A$	+1	+	0	+	+1	+	0	+
(7) $\sin^2 A$	0	+	+1	+	0	+	+1	+

	0^0	45^0	90^0	135^0	180^0	225^0	270^0	315^0
(8) $\sin A \cdot \cos A$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$
(9) $\sin A + \cos A$	+1	$+\sqrt{2}$	+1	0	−1	$-\sqrt{2}$	−1	0
(10) $\tan A + \cot A$	∞	+2	∞	−2	∞	+2	∞	−2
(11) $\sin A - \cos A$	−1	0	+1	$+\sqrt{2}$	+1	0	−1	$-\sqrt{2}$

The following figures exhibit the changes in the sign and magnitude of $\sin\theta$ (fig. i.), $\cos\theta$ (fig. ii.), and $\tan\theta$ (fig. iii.). The measure of the distance from 0 along the line $0X$ = the circular measure of the angle; the vertical distance from $0X$ measures the Ratio.

Fig. i. The Curve of the Sine.

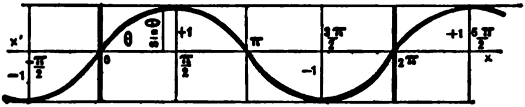


Fig. ii. The Curve of the Cosine.

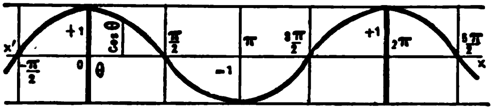
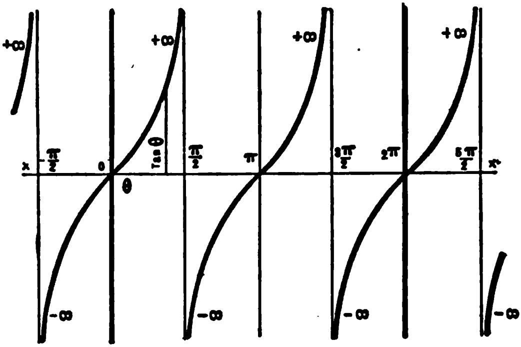


Fig. iii. The Curve of the Tangent.



XXXI. Page 112.

(1) (i) 30°, 150°, -210°, -330°. (ii) 45°, 135°, -225°, -315°.
(iii) 60°, 120°, -240°, -300°. (iv) -30°, -150°, 210°, 330°.

(2) (i) 20°, 160°, 380°, 520°. (ii) $\frac{5\pi}{4}$, $\frac{7\pi}{4}$, $\frac{13\pi}{4}$, $\frac{15\pi}{4}$.

(iii) $\frac{8\pi}{7}$, $\frac{13\pi}{7}$, $\frac{22\pi}{7}$, $\frac{27\pi}{7}$.

(3) (i) $\theta = n\pi + (-1)^n\left(-\frac{\pi}{6}\right)$. (ii) $\theta = n\pi + (-1)^n\frac{\pi}{6}$.

(iii) $n\pi \pm \frac{\pi}{4}$.

XXXII. Page 115.

(1) (i) $\theta=2n\pi\pm\frac{1}{3}\pi$. (ii) $\theta=n\pi+\frac{1}{4}\pi$. (iii) $\theta=n\pi-\frac{1}{4}\pi$.
(iv) $\theta=n\pi-\frac{1}{3}\pi$. (v) $\theta=2n\pi\pm\frac{4}{5}\pi$. (vi) $\theta=n\pi+\frac{3}{4}\pi$.
(3) The tangent. (4) No.
(5) (i) 60^0, -60^0, 300^0, -300^0. (ii) 45^0, 225^0, 405^0, -135^0.
(iii) -45^0, -225^0, 135^0, 315^0. (iv) -60^0, -240^0, 120^0, 300^0.
(v) 144^0, -144^0, 216^0, -216^0. (vi) 135^0, -45^0, -225^0, 315^0,

XXXIV. Page 120.

(5) 1, $\frac{24}{25}$. (6) $\frac{16}{65}$, $\frac{33}{65}$.

XXXVIII. Page 131.

(1) $\sin(\theta+\phi)+\sin(\theta-\phi)$. (2) $\cos(\alpha-\beta)+\cos(\alpha+\beta)$.
(3) $\sin(2\alpha+3\beta)+\sin(2\alpha-3\beta)$. (4) $\cos 2\alpha+\cos 2\beta$.
(5) $\sin 8\theta-\sin 2\theta$. (6) $\cos\theta+\cos 2\theta$. (7) $\frac{1}{2}(\cos 3\theta-\cos 5\theta)$.
(8) $\frac{1}{2}(\sin 4\theta-\sin\theta)$. (9) $\sin 60^0+\sin 40^0$.
(10) $\frac{1}{2}(\sin 60^0-\sin 30^0)$. (11) $2\cos 3\theta\cos 2\theta$.
(12) $-\cos 4\theta\sin 2\theta$. (13) $4\cos^2\frac{\theta}{2}\sin 2\theta$.

XXXIX. Page 132.

(4) $\tan 2a$. (5) $\tan x$. (7) $-2\cot 2x$. (8) $\tan^2 A$.
(12) $\sin 2\theta=1$, $\therefore\ \theta=45^0$; $\sin 2\phi=\frac{1}{2}$, $\therefore\ \phi=15^0$.

XLV. Page 147.

(2) $\sin 2A=\frac{4}{5}$, $\sin 3A=-\frac{11}{5}\sqrt{5}$.
(3) $\sin 2\theta=-\frac{1}{8}\sqrt{15}$, $\sin 3\theta=\frac{3}{16}\sqrt{15}$, $\cos 3\theta=-\frac{11}{16}$; in the second.

XLVI. Page 149.

(3) $\sin\frac{A}{2}=+\sqrt{\frac{1-\cos A}{2}}$. (5) $\sin\frac{A}{2}=+\sqrt{\frac{1-\cos A}{2}}$.

XLVII. Page 154.

(1) (i) $+$, $-$. (ii) $-$, $+$. (iii) $-$, $-$. (iv) $+$, $-$. (v) $+$, $-$.
(vi) $+$, $+$. (vii) $-$, $-$. (viii) $-$, $-$.
(3) $\sin 9^0=\frac{1}{4}\{\sqrt{(3+\sqrt{5})}-\sqrt{(5-\sqrt{5})}\}$,
$\cos 9^0=\frac{1}{4}\{\sqrt{(3+\sqrt{5})}+\sqrt{(5-\sqrt{5})}\}$,
$\sin 81^0=\cos 9^0$, $\cos 189^0=-\cos 9^0$, $\tan 202\frac{1}{2}^0=\sqrt{2}-1$,
$\tan 97\frac{1}{2}^0=-(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$.
(7) Between $(2n\times 180^0-45^0)$ and $(2n\times 180^0+45^0)$.

XLVIII. Page 157.

(1) $\theta = -\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{6}$. (2) $\theta = -\frac{\pi}{3} + n\pi + (-1)^n \frac{\pi}{6}$.

(3) $\theta = -\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{8}$. (4) $\theta = +\frac{\pi}{4} + n\pi + (-1)^n \frac{\pi}{4}$.

(5) $\theta = +\frac{\pi}{4} + 2n\pi \pm \frac{\pi}{4}$. (6) $\theta = \frac{\pi}{6} + n\pi \pm \frac{\pi}{4}$.

(7) $x = -68^0\,12' + n \times 180^0 + (-1)^n\,(21^0\,48')$.

(8) $x = -69^0\,26'\,30'' + 2n \times 180^0 \pm (69^0\,26'\,30'')$.

(9) $x = 75^0\,4' + n \times 180^0 + (-1)^n\,(14^0\,56')$.

(10) $x = -a - \frac{\pi}{4} + 2n\pi \pm \beta$.

L. Pages 159, 160.

(7) $\pm(\sqrt{3}+1)$. (8) $\pm\sqrt{2}$. (9) ± 1.

(13) $\frac{1}{6}(2n+1)\pi$; or, $\frac{1}{3}(3n \pm 1)\pi$. (14) $n\pi \pm \frac{\pi}{3}$; or $\frac{1}{6}(2n+1)\pi$.

(15) $\frac{1}{6}(2n+1)\pi$; or $\frac{n\pi}{3} + (-1)^n \frac{\pi}{12}$.

(16) $(2n+1)\pi$; or, $\cos^{-1}\frac{a-b}{a}$. (17) $\frac{n\pi}{m}$; or $\frac{2r+1}{m \pm n}.\pi$.

(18) Both equations are satisfied if $(x^2 - y^2) =$ an odd integer; or, if $(x+y)^2 = 2n$, and $(x-y)^2 = 4m - 2n$, m and n being integers.

(19) (i) $2\sin\theta.\cos\theta = \sin 2\theta$, and therefore it goes through the same changes as $\sin\theta$, while θ changes from 0 to 2π. (ii) $\cos^2\theta - \sin^2\theta = \cos 2\theta$. (iii) $\sin 3\theta$ goes through the same changes as $\sin\theta$, while θ goes from 0 to 3π. (iv) $\cot 2A$; compare with $\cot A$. (v) $\sin(\theta + a)$ goes through the same changes while θ goes from 0 to π, as $\sin\theta$ goes through while θ goes from a to $\pi + a$. (vi) $\cos(2\theta - a)$ goes through the same changes while θ goes from 0 to π, as $\cos 2\theta$ goes through as θ changes from $-\frac{a}{2}$ to $\pi - \frac{a}{2}$. (See Ans. to XXVII.)

(20) They are the solutions of the equations $\sin\theta = \sin a$, and $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(\frac{\pi}{2} - a\right)$; and we know that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$.

(21) They are the solutions of the equations $\sin\left(\theta + \frac{\pi}{4}\right) = \sin\frac{\pi}{6}$, and $\cos\left(\theta - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}$; also $\cos\left(\theta - \frac{\pi}{4}\right) = \sin\left(\theta + \frac{\pi}{4}\right)$, $\sin\frac{\pi}{6} = \cos\frac{\pi}{3}$.

LI. Page 162.

(1) (i) a^{2h+3k}, (ii) a^{4h-5k}, (iii) $a^{\frac{4h}{3}+\frac{5k}{3}}$. (iv) $a^{\frac{5h}{2}+\frac{3k}{2}}$.
(2) (i) 5·4690116. (ii) 10·6243928. (iii) 13·7509366.
(iv) ·8853661. (v) 1·7968680. (vi) 8·9699598.
(vii) 2·7345058.
(3) $2^3, 2^5, 2^{-1}, 2^{-4}, 2^{-3}, 2^7$. (4) $3^2, 3^4, 3^{-1}, 3^{-3}, 3^{-2}, 3^{-4}$.

LII. Page 163.

(1) ·60206, ·9542426, ·90309, ·7781513, 1·20412, 1·690196.
(2) 1·146128, 1·20412, 1·2552726, 1·3802113, 1·4313639, 1·6232493.
(3) 1, ·69897, 1·1760913, 1·39794, 1·4771213, 1·5440680.
(4) 1·5563026, 1·60206, 1·6812413, 1·69897, 2·30103, 3.
(5) 7·201593, 3·858708. (6) ·7545579, 2·989843.
(7) 1·4532. (8) 2408·6. (9) (i) 4·5868. (ii) ·93646.
(10) 3·9549. (11) 40975·3 sq. ft. (12) 34·925 in.
(13) 3·2617 in. (14) 110115 cub. yds.

LIII. Page 165.

(1) 3, $\frac{10}{3}$, $\frac{1}{4}$, $\frac{2}{3}$, $-\frac{5}{2}$. (2) 3, 6, −1, −3, −6, 2.
(3) 2, 4, −1, −3, −2, −4. (4) $\frac{3}{2}$, $\frac{2}{3}$, $-\frac{1}{6}$, −1.
(5) 3, −1, 5, −2, 3, −3. (6) $\frac{4}{3}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$.
(7) ·7781513, 1·6232493, 1·20412.
(8) 1·6901960, 1·5563026, 1·7993406.
(9) 2·30103, 2·7781513, 1·845098.
(10) ·69897, 5228787, 1·69897.
(11) 1·544068, 2·1760913, −1+·30103.
(12) ·5440680, ·8627278, −2+·9084852.

LIV. Pages 168, 169.

(1) 4, 2, 0, 5, 1. (2) −2, −5, −1, −3.
(3) 3, −1, 0, 1, 0, −7. (4) 4, 1, 6, 3.
(5) the second decimal place, the first dec. pl., the sixth dec. pl.
(6) ten thousands, units, hundreds, third dec. pl., first dec. pl., units.
(7) 10, 4, 25, 31. (8) 9, 11, 85, 4, 9, 6.
(9) units, fourth dec. pl., thousands, seventh dec. pl., second dec. pl.
(10) tenth integral pl., twelfth dec. pl., fifth dec. pl., units, twelfth dec. pl., first dec. pl.

LV. Pages 171, 172.

(1) 2·8901023, ·8901023, $\bar{4}$·8901023, 5·8901023.

(2) 6·7714552, ·7714552, $\bar{4}$·7714552, 2·7714552, 3·7714552.

(3) ·27724... (4) ·00001638... (5) ·77448... (6) ·005968...

LVI. Page 174.

(1) Divide each log by 3. (2) Multiply each log by 2.

(3) Divide each log by ·30103.

(4) Multiply each log by ·4771213.

(5) Divide each log by ·4771213. (6) 3·32190...

(7) 1·183... (8) 1·10730... ·66438...

LVII. Pages 175, 176.

(1) 3, 0, $\frac{1}{3}$, 0, $\frac{7}{5}$. (4) $\bar{1}$·8121177, 55.

(5) ·51375. (6) 7, 4, 3, 3.

(7) (i) $x = \dfrac{2 \log 7}{\log 2 + 4 \log 7}$. (ii) $x = \dfrac{7 \log 2 + 4 \log 7}{2 \log 3 + \log 7}$.

(iii) $x = \dfrac{2 \log 7}{2 \log 2 + \log 3}$. (iv) $x = \dfrac{4 (\log 3 + \log 7)}{8 \log 2 + 3 (\log 3 + \log 7)}$.

(8) $2 + \dfrac{1}{\log_{10} 7}$. (9) $\frac{3}{2} + \dfrac{1}{\log_{10} 3}$. (10) $\dfrac{1}{1 - \log_{10} 2}$.

(11) 0, $\dfrac{1}{b+1}$, $\dfrac{3a}{2b+2}$, $\dfrac{2}{b+1}$, $\dfrac{b}{b+1}$, $\dfrac{3a+2}{2b+2}$, $\dfrac{bc}{b+1}$.

(12) $63 - 31 = 32$. (13) $(a^{11} - a^{10})$ integers. (14) 1·9485 nearly.

(19) 2·53855. (20) 4·53999. (21) 167 years.

LVIII. Pages 181, 182.

(1) ·8839066. (2) 2·7513738. (3) $\bar{4}$·9413333.

(4) 6·8086920. (5) ·5710750. (6) 3·70404.

(7) 45740·26. (8) 2492837. (9) ·000439658. (10) 5·689158.

LIX. Pages 184, 185.

(1) ·6737652. (2) ·6737652 (3) ·9306572.

(4) 41° 48′ 37″. (5) 70° 31′ 43·6″. (6) 75° 31′ 21″.

(7) 9·8515594. (8) 9·7114477. (9) 10·1338768.

(10) 85° 4′ 23″. (11) 23° 16′ 27·5″. (12) 21° 56′ 41″

LX. Pages 188, 189.

(1) 34° 19′ 31·8″. (2) 1498·2 ft. (3) 45° 36′ 56″.

(4) 5293·4 ft., 6982·3 ft. (5) 576·2 chains.

(6) 4729 chains. (7) 3666·8 feet. (8) 42° 15′, 11444 chains.

LXI. Page 190.

(1) 3842·9 ft. (2) 281·74 ft. (3) 115·3 ft. (4) 285·6 ft.

(5) 58° 17′ 24″, 31° 42′ 36″. (6) 656·1 chains, 41° 17′ 12″.

(7) 81 ft. (8) 1942 ft. (9) 646·7 miles. (10) 1000 ft.

LXVI. Pages 208, 209.

(1) 41° 16′ 51·5″. (2) 73° 32′ 12″, 62° 46′ 18″.

(3) 29° 17′ 16″, 31° 55′ 31″. (4) 64° 31′ 58″.

(5) 73°, 23′ 54·4″. (6) 41° 24′ 34·6″. (7) 82° 49′ 9″.

(8) 75°, 60°, 45°. (9) 135°, 30°, 15°.

LXVII. Page 211.

(1) 313·46 yds. (2) 28·87 inches, 31·43 inches.

(3) 1192·55 yds. (4) 22·415 ft.

(5) 24·995 = 25 ft. nearly, 17·559 ft., 65° 59′ 42″.

LXVIII. Pages 213, 214.

(1) 108° 36′ 30″, 31° 23′ 30″. (2) 93° 11′ 49″, 36° 48′ 11″.

(3) 57″ 27′ 25·4″, 62° 32′ 34·6″. (4) 64° 26′ 47″, 37°, 7′ 13″.

(5) 72° 12′ 59″. (6) 20·5 chains. (7) 122·7.

(8) 71° 13′ 50″, 82° 16′ 10″.

LXIX. Pages 218, 219.

(1) $A = 91^\circ 18' 21''$, $C = 88^\circ 41' 39''$; or $A = 128^\circ 41' 39''$, $C = 11^\circ 18' 21''$.

(2) $B = 70^\circ 0' 56''$, $C = 59^\circ 59' 4''$; or, $B = 109^\circ 59' 4''$, $C = 20^\circ 0' 56''$.

(3) $B = 38^\circ 38' 24''$, $C = 91^\circ 21' 36''$, $c = 155{\cdot}3$. (4) 61° 16′ 10″.

(5) $A = 72^\circ 4' 48''$, $B = 41^\circ 56' 12''$; or, $A = 107^\circ 55' 12''$,
$B = 6^\circ 5' 48''$, $b = 17{\cdot}56$. (6) β is ambiguous; 60·3898 ft.

LXX. Page 220.

The angles are given correct to the nearest second.

(1) $28^0\,35'\,39''$. (2) $104^0\,44'\,39''$. (3) $32^0\,20'\,48''$.
(4) $43^0\,40'$. (5) $128^0\,23'\,13''$. (6) 106531 ft.
(7) 3437·6 yds. (8) 1728·2 chains. (9) 25376 yards.
(10) $A=66^0\,27'\,48'', B=12^0\,55'\,12''$. (11) $A=92^0\,12'\,53'', B=35^0\,37'\,7''$.
(12) $B=29^0\,1'\,40'', C=74^0\,55'\,50''$. (13) $B=70^0\,35'\,24''$; or, $109^0\,24'\,36''$.
(14) $B=51^0\,56'\,17''$; or, $128^0\,3'\,43''$. (15) $B=62^0\,6'\,10''$; or, $117^0\,53'\,50''$.
(16) Very nearly 90^0. (17) 1319·6 yds.

LXXII. Page 229.

(15) To find the point E in an unlimited straight line CE at which a finite straight line AB subtends the greatest angle, a circle must be described passing through A and B, and touching the line CE in the point E. In (15) the centre of the circle lies vertically above E, and in the horizontal line through the middle point of AB.

LXXIII. Page 239.

(1) (i) 10 sq. ft. (ii) 43·3 sq. in. (iii) 148·13 sq. yds.
(iv) 84 sq. chains = 8·4 acres (v) 100 sq. ft.
(vi) 151872 sq. yds.

(2) 4, 10½, 12, 14 ft. (4) 8⅛ ft.

LXXVI. Pages 253—256.

(10) $-(x+y+z)(y+z-x)(z+x-y)(x+y-z)$.

(11) (i) $\frac{1}{18}\{6n\pi-2\pi+(-1)^n\pi\}$. (ii) $\dfrac{4r\pi+\pi}{2(m\pm n)}$.

(iii) $x=\dfrac{a-\beta}{2}+\cot^{-1}\left\{\tan\dfrac{a+\beta}{2}\cdot\dfrac{m\sin\beta+n\sin a}{n\sin a-m\sin\beta}\right\}$.

(iv) $\dfrac{2r\pi+\pi}{2(m+n)}$. (v) $\sin 2\theta\,(3\cos 2\theta-1)(2\cos 2\theta+1)=0$.

(vi) $\sin 4\theta\,.\cos\dfrac{5\theta}{2}\,.\sin\dfrac{3\theta}{2}=0$. (vii) $\sin 8\theta\,.\sin 4\theta=0$.

(21) 223·17.

APPENDIX.

PART I.

(1) Simplify the formulæ

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \qquad \cos \tfrac{1}{2} A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

in the case of an equilateral triangle.

(2) The sides of a triangle are as $2 : \sqrt{6} : 1+\sqrt{3}$, find the angles.

(3) The sides of a triangle are as $4,\ 2\sqrt{2},\ 2(\sqrt{3}-1)$, find the angles.

(4) Given $C=120^0$, $c=\sqrt{19}$, $a=2$, find b.

(5) Given $A=60^0$, $b=4\sqrt{7}$, $c=6^0\sqrt{7}$, find a.

(6) Given $A=45^0$, $B=60^0$ and $a=2$, find c.

(7) The sides of a triangle are as $7 : 8 : 13$, find the greatest angle.

(8) The sides of a triangle are $1,\ 2,\ \sqrt{7}$, find the greatest angle.

(9) The sides of a triangle are as $a : b : \sqrt{(a^2+ab-b^2)}$, find the greatest angle.

(10) When $a : b : c$ as $3 : 4 : 5$, find the greatest and least angles; given $\cos 36^0\, 52' = \cdot 8$.

(11) If $a=5$ miles, $b=6$ miles, $c=10$ miles, find the greatest angle. [$\cos 49^0\, 33' = \cdot 65$.]

(12) If $a=4$, $b=5$, $c=8$, find C; given that $\cos 54^0 54'=\cdot575$.

(13) The sides of a triangle are 7, 8, 13, find the greatest angle.

(14) Given $C=18^0$, $a=\sqrt{5}+1$, $c=\sqrt{5}-1$, find the other angles.

(15) If $b=3$, $C=120^0$, $c=\sqrt{13}$, find a and the sines of the other angles.

(16) Given $A=105^0$, $B=45^0$, $c=\sqrt{2}$, solve the triangle.

(17) Given $B=75^0$, $C=30^0$, $c=\sqrt{8}$, solve the triangle.

(18) Given $B=45^0$, $c=\sqrt{75}$, $b=\sqrt{50}$, solve the triangle.

(19) Given $B=30^0$, $c=150$, $b=50\sqrt{3}$, show that of the two triangles which satisfy the data one will be isosceles and the other right-angled. Find the third side in the greatest of these triangles.

(20) Is the solution ambiguous when $B=30^0$, $c=150$, $b=75$?

(21) If the angles adjacent to the base of a triangle are $22\frac{1}{2}^0$ and $112\frac{1}{2}^0$, show that the perpendicular altitude will be half the base.

(22) If $a=2$, $b=4-2\sqrt{3}$, $c=\sqrt{6}(\sqrt{3}-1)$, solve the triangle.

(23) If $A=9^0$, $B=45^0$, $b=\sqrt{6}$, find c.

(24) Given $B=15^0$, $b=\sqrt{3}-1$, $c=\sqrt{3}+1$, solve the triangle.

(25) Given $\sin B=\cdot25$, $a=5$, $b=2\cdot5$, find A. Draw a figure to explain the result.

(26) Given $C=15^0$, $c=4$, $a=4+\sqrt{48}$, solve the triangle.

(27) Two sides of a triangle are $3\sqrt{6}$ yards and $3(\sqrt{3}+1)$ yards, and the included angle 45^0, solve the triangle.

(28) If $C=30^0$, $b=100$, $c=45$, is the triangle ambiguous?

(29) Prove that if $A=45^0$ and $B=60^0$ then $2c=a(1+\sqrt{3})$.

(30) The cosines of two of the angles of a triangle are $\frac{1}{2}$ and $\frac{3}{5}$, find the ratio of the sides.

ANSWERS TO APPENDIX.

(1) $\cos A = \frac{1}{2}$, $\cos \frac{1}{2}A = \frac{1}{2}\sqrt{3}$. (2) 45°, 60°, 75°.
(3) 130°, 30°, 15°. (4) 3. (5) 14. (6) $1+\sqrt{3}$. (7) 120°.
(8) 120°. (9) 120°. (10) 90°, $36^\circ\,52'$. (11) $130^\circ\,27'$. (12) $125^\circ\,6'$.
(13) 120°. (14) $A=54^\circ$ or 126°, $B=108^\circ$ or 36°. (15) $a=1$.
(16) $C=30^\circ$, $a=\sqrt{3}+1$, $b=2$. (17) $A=75^\circ$, $a=b=\sqrt{3}+1$.
(18) $C=60^\circ$ or 120°. (19) $100\sqrt{3}$. (20) no.
(22) $A=105^\circ$, $C=60^\circ$, $B=15^\circ$. (23) $\frac{1}{2}\sqrt{3}(\sqrt{5}+1)$.
(24) $A=90^\circ$ or 60°, $C=75^\circ$ or 105°, $a=2\sqrt{2}$ or $\sqrt{6}$. (25) 30°
or 150°. (26) $A=45^\circ$ or 135°, $B=30^\circ$ or 120°, $b=\sqrt{2}(1+\sqrt{3})$
or $\sqrt{6}(1+\sqrt{3})$. (27) 60°, 75°, 6 yds. (28) It is impossible.
(30) $15 : 8\sqrt{3} : 4\sqrt{5}+6$.

CAMBRIDGE: PRINTED BY C. J. CLAY, M.A. & SON, AT THE UNIVERSITY PRESS.

January, 1885.

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